STARK-WANNIER TYPE OPERATORS WITH PURELY SINGULAR SPECTRUM

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Abstract. We consider the one-dimensional Stark-Wannier type operators

$$H = -\frac{d^2}{dx^2} - Fx - q(x) + v(x), \quad F > 0,$$

where q is a smooth function slowly growing at infinity and v is periodic, $v \in L_1(\mathbb{T})$, with the Fourier coefficients of the form $(\ln |n|)^{-\beta}$, $0 < \beta < \frac{1}{2}$, as $n \to \infty$. We show that for suitable q and F the spectrum of the corresponding operator is purely singular continuous. This proves the sharpness of the a.c. spectrum stability result obtained in [16].

1. INTRODUCTION

In this paper we consider the following operators on $\mathbb{R}$

$$H = -\frac{d^2}{dx^2} - Fx - q(x) + v(x), \quad F > 0, \tag{1.1}$$

where q is a smooth function slowly growing at infinity:

$$|q^{(k)}(x)| \le c < x >^{\alpha - k}, \quad \alpha < 1, \quad k = 0, 1, 2,$$

$< x >= (1 + x^2)^{1/2}$, and v is periodic, $v(x + 1) = v(x)$, $v \in L_1(\mathbb{T})$,

$$\int_0^1 dx v(x) = 0. \tag{1.2}$$

The spectral properties of this model have been extensively discussed in both mathematical and physical literature, see [1-5, 7, 10, 15-17] and references therein. If $v = 0$ the spectrum of H is purely absolutely continuous:

$$\sigma(H) = \sigma_{ac}(H) = \mathbb{R}. \tag{1.3}$$

It is known that the spectrum of (1.1) remains absolutely continuous and covers the whole axis under rather weak smoothness requirements on the potential v. In fact, (1.3) can be proved for $v \in H^s(\mathbb{T}),\ s > 0$, see [7, 17]. On the other hand there are examples of very singular periodic perturbations such as an array of δ' potentials for which the spectrum has no a.c. component or even is pure point, see [2,3, 15]. It was argued by Ao [1] that the spectral nature depends on the gap structure of

the periodic perturbation. He conjectured that if the size of the n-th gap behaves as $n^{-\alpha}$ the spectrum is pure point for $\alpha < 0$ (at least for "non-resonant" F) and continuous for $\alpha > 0$. In the critical case $\alpha = 0$, which corresponds to a comb of δ function potentials, a transition from pure point to continuous spectrum is expected as F grows (see also [9, 14] for related phenomena in the random setting). Furthermore, the spectral nature in this critical case seems to depend also on the number theoretical properties of F [7]. In [16] the following sufficient condition for the stability of the a.c. spectrum was established.

Theorem 1.1. *Suppose that*

$$v \in L_1(\mathbb{T}) \cap H^{-1/2}(\mathbb{T}). \tag{1.4}$$

Then, an essential support of the absolutely continuous spectrum of the operator H coincides with the whole axis.

Condition (1.4) corresponds to $\alpha > 0$ in Ao language. Some intermediate results were obtained in [17].

Remark. If $v \in L_1(\mathbb{T})$ then the essential spectrum of H, $\sigma_{ess}(H)$, fills up the real axis:

$$\sigma_{ess}(H) = \mathbb{R}. \tag{1.5}$$

The proof of this fact is a simple compactness type argument, for the sake of completeness we outline it in appendix 3.

The goal of the present paper is to show that the result of theorem 1.1 is optimal. We consider operator (1.1) with $q(x) = \kappa \ln < x >$:

$$H = -\frac{d^2}{dx^2} - Fx - \kappa \ln < x > + v(x), \quad F > 0. \tag{1.6}$$

To avoid cumbersome general conditions we consider the case where the Fourier coefficients of v, $\hat{v}(n) = \int_0^1 dx e^{i2\pi nx} v(x)$, satisfy for some $n_0 \geq 2$

$$\hat{v}(n) = v_0 (\ln n)^{-\beta}, \quad n \geq n_0, \ v_0 \neq 0, \ 0 < \beta < \frac{1}{2}. \tag{1.7}$$

Remark that (1.7) implies

$$v \in C^\infty(\mathbb{R} \setminus \mathbb{Z}) \cap L_{1,loc}(\mathbb{R}).$$

Our main result is the following theorem.

Theorem 1.2. *For $\frac{\pi^2}{F} \in \mathbb{Q}$ and $\kappa \neq 0$ the spectrum of* (1.6) *is purely singular continuous.*

Remarks.

1. If $\frac{\pi^2}{F}$ is rational and $\kappa = 0$ the spectrum of (1.6) is absolutely continuous except, may be some discrete set of the eigenvalues, see [7] and also subsection 2.1.

2. With some extra efforts the methods of the present paper can be made work also for $\beta = \frac{1}{2}$

The rest of the paper is devoted to the proof of theorem 1.2. The basis for our analysis is the asymptotic constructions of [7] that we combine with the ideas of the works [6, 8, 14]. In [7] operator (1.1) with $q=0$ and

$$v(x) = V\sum_l \delta(x-l)$$

was considered. It was shown that the spectral properties of this model can be characterized by the asymptotic behavior as $l\to+\infty$ of the solutions of the following discrete system

$$\psi(l+1) = W(l)\psi(l), \quad \psi(l)\in\mathbb{C}^2, \tag{1.8}$$

where

$$W(l) = e^{i\Gamma(l)\sigma_3}S(l)e^{-i\Gamma(l)\sigma_3}, \quad \det S(l) = 1, \tag{1.9}$$

$$\Gamma(l) = \frac{(\pi l)^3}{3F} + \frac{\pi l(E-V)}{F}, \quad S(l) = \begin{pmatrix} 1+dl^{-1} & rl^{-1/2} \\ \bar{r}l^{-1/2} & 1+dl^{-1}\end{pmatrix},$$

$d\in\mathbb{R}$, $r\in\mathbb{C}$.

When applied to (1.6) (with $\frac{\pi^2}{F}\in\mathbb{Q}$ and $\kappa\neq 0$) the constructions of [7] lead to model (1.8), (1.9) with

$$\Gamma(l)\sim\rho(E)\Lambda^l, \quad S(l) = I+O(l^{-\beta}), \quad \Lambda>1.$$

This system is close to the case of discrete Schrödinger operators with strongly mixing decaying potentials and the techniques of [6, 8, 14] can be applied.

2. Reduction to a strongly mixing model.

2.1. Some preliminary reductions.

2.1.1. Prüfer type coordinates.

To derive the desired spectral properties we study the solutions of the equation

$$-\psi'' - Fx\psi - q(x)\psi + v(x)\psi = E\psi, \quad E\in\mathbb{R}, \tag{2.1}$$

the link between the behavior of solutions and spectral results being provided by Gilbert-Pearson subordinacy theory [12, 13]. As soon as the a.c. spectrum is concerned the whole line operator can be replaced by a right half -line operator with some self-adjoint boundary conditions in a point $x=R$. Indeed, since $-\frac{d^2}{dx^2}+v$ is bounded from below and $-Fx-q\to+\infty$ as $x\to-\infty$, the spectrum of the left half-line operator is discrete. So, the absolutely continuous parts of the whole line operator and the right half-line operator are unitarily equivalent and by subordinacy theory to prove theorem 1.2 it is sufficient to show that

(i) for a.e. $E\in\mathbb{R}$ there exist a solution subordinate on the right;

(ii) for any $E\in\mathbb{R}$ there is no solution which is in L_2 on the right.

Recall that a real solution $\psi_1(x,E)$ of (2.1) is called subordinate on the right if for any other linearly independent solution $\psi(x,E)$ one has

$$\lim_{N\to+\infty}\frac{\int_0^N dx|\psi_1(x,E)|^2}{\int_0^N dx|\psi(x,E)|^2} = 0.$$

The main part of the paper is devoted to the proof of part (i), (ii) being obtained as a simple by-product of our constructions, see proposition 2.1.

To study the asymptotic behavior of the solutions of (2.1) we employ Prüfer type transformation which is known to be extremely useful in the cases of small or decaying randomness, see [6, 8, 11, 14].

First, we perform a Liouville transformation in (2.1), setting for x sufficiently large

$$\psi(x, E) = p^{-1/2} Q(\xi(x)), \quad p = (Fx + q + E)^{1/2}, \quad \xi'(x) = p(x, E),$$

$$\xi = \frac{2}{3F}(Fx + q + E)^{3/2} - \frac{2\kappa}{F}(Fx + q + E)^{1/2} + O(x^{-1/2} \ln x), \quad x \to +\infty.$$

The resulting function Q satisfies the Schrödinger equation

$$-Q_{\xi\xi} - Q + \frac{V}{p} Q = 0, \tag{2.2}$$

$$V = vp^{-1} + \frac{1}{4} q'' p^{-3} - \frac{5}{16}(F + q')^2 p^{-5}.$$

Let us further apply a Prüfer type transformation:

$$Q = R \sin\theta,$$

$$Q_\xi = R \cos\theta.$$

Then R, θ satisfy

$$\frac{d}{dx} \ln R = \frac{1}{2} V \sin 2\theta,$$

$$\frac{d}{dx}\theta = p - V \sin^2\theta. \tag{2.3}$$

It is not difficult to check that

$$\int_{N_1}^{N_2} dx \psi^2 \leq \int_{N_1}^{N_2} dx p^{-1} R^2 \leq C \int_{N_1}^{N_2} dx \psi^2, \tag{2.4}$$

provided $N_2 \geq N_1$ are sufficiently large: $N_1 \geq C$.

The constants C here and below are uniform with respect to E in compact subsets of $\mathbb{R}$ but may depend on q and v.

2.1.2. Reduction to a discreet system. First we are going to analyse R along a sequence $x = x_l$, where $x_l = [\tilde{x}_l] - 1/2$, $\tilde{x}_l$ being defined by the relation

$$F\tilde{x}_l + q(\tilde{x}_l) + E = \pi^2 \left(l - \frac{1}{2}\right)^2.$$

We start by a sequences of auxiliary estimates.

Lemma 2.1. *For $x \in [x_l, x_{l+1}]$, $v \in L_1(\mathbb{T})$ and any $0 < \nu < \frac{1}{2}$ one has the following inequalities*

$$|\int_{x_l}^{x} dy \chi_l(y) e^{2i\xi(y)} v(y)| \le c(l^{1/2}|\hat{v}_l| + \|v\|_{L_1(\mathbb{T})}),$$

$$|\int_{x_l}^{x} dy e^{2i\xi(y)} v(y)(1-\chi_l(y))| \le c(l^{\nu}|\hat{v}_l| + \|v\|_{L_1(\mathbb{T})}).$$

Here $\chi_l(x) = \chi(l^{-1+\nu}(x - X_l))$, $\chi \in C_0^\infty$, $\chi(s) = \chi(-s)$,

$$\chi(s) = \begin{cases} 1, & \text{if } |s| \le 1, \\ 0 & \text{if } |s| \ge 2, \end{cases}$$

X_l solves the equation

$$FX_l + q(X_l) + E = \pi^2 l^2.$$

See appendix 1 for the proof.

It follows immediately from lemma 2.1 and (2.2) that for $x \in [x_l, x_{l+1}]$, and any $g \in L_1(\mathbb{T})$

$$\int_{x_l}^{x} dy e^{2i\theta(y)} g(y) = e^{2i\varphi(x_l)} \int_{x_l}^{x} dy e^{2i\xi(y)} g(y)$$

$$-2i \int_{x_l}^{x} dy e^{2i\varphi(y)} V(y) \sin^2\theta(y) \int_{y}^{x} ds e^{2i\xi(s)} g(s) = O(l^{1/2})\|g\|_{L_1(\mathbb{T})}, \quad \varphi = \theta - \xi. \tag{2.5}$$

Here and below the constants in $O(\cdot)$ are independent of the choice of initial conditions in (2.1), uniform with respect to E in compact subsets of $\mathbb{R}$, but may depend on q and v.

Consider the expression $\ln \frac{R(x)}{R(x_l)}$. By (2.2), (2.3) and lemma 2.1,

$$\ln \frac{R(x)}{R(x_l)} = \frac{1}{2} \operatorname{im} \left(\int_{x_l}^{x} dy e^{2i\theta} v p^{-1} \right) + O(l^{-4}) = \frac{1}{2\pi l} \operatorname{im} \left(e^{2i\varphi(x_l)} \int_{x_l}^{x} dy e^{2i\xi} v \right)$$

$$-\frac{1}{(\pi l)^2} \operatorname{re} \left(\int_{x_l}^{x} dy e^{2i\varphi(y)} v(y) \sin^2\theta(y) \int_{y}^{x} ds e^{2i\xi(s)} v(s) \right) + O(l^{-3/2}), \tag{2.6}$$

which means in particular that

$$\ln \frac{R(x)}{R(x_l)} = O(l^{-1/2}). \tag{2.7}$$

Therefore, if we control $R(x_l, E)$ we will have sufficient control for all x.

It follows from lemma 2.1 and (1.2), (2.5) that

$$\int^{x} dy e^{2i\varphi(y)} v(y) \int_{y}^{x} ds e^{2i\xi(s)} v(s)$$

$$= \int_{x_l}^{x} dy e^{2i\theta(y)} v(y) \int_{x_l}^{y} ds v(s) + O(l^{1/2}) = O(l^{1/2}), \tag{2.8}$$

$$\mathrm{re} \int_{x_l}^{x} dy e^{-2i\xi(y)} v(y) \int_y^x ds e^{2i\xi(s)} v(s) = \frac{1}{2} \left| \int_{x_l}^{x} dy e^{2i\xi(y)} v(y) \right|^2,$$

$$\int_{x_l}^{x} dy e^{2i\varphi(y)+2i\theta(y)} v(y) \int_y^x ds e^{2i\xi(s)} v(s) = \frac{1}{2} e^{4i\varphi(x_l)} \left(\int_{x_l}^{x} dy e^{2i\xi(y)} v(y) \right)^2 + O(l^{1-\gamma}).$$

We use γ as a general notations for universal positive constants, they may change from line to line. Combining (2.6), (2.8) one obtains

$$\ln \frac{R(x_{l+1})}{R(x_l)} = \frac{1}{2\pi l} \mathrm{im} \left(e^{2i\varphi(x_l)} \int_{x_l}^{x_{l+1}} dy e^{2i\xi(y)} v(y) \right) + \frac{1}{8\pi^2 l^2} \left| \int_{x_l}^{x_{l+1}} dy e^{2i\xi(y)} v(y) \right|^2$$

$$+ \frac{1}{8\pi^2 l^2} \mathrm{re} \left[e^{4i\varphi(x_l)} \left(\int_{x_l}^{x_{l+1}} dy e^{2i\xi(y)} v(y) \right)^2 \right] + O(l^{-1-\gamma}). \tag{2.9}$$

In a similar way,

$$\varphi(x_{l+1}) - \varphi(x_l) = \frac{1}{2} \mathrm{re} \left(\int_{x_l}^{x_{l+1}} dy e^{2i\theta} v p^{-1} \right) + O(l^{-2})$$

$$= \frac{1}{2\pi l} \mathrm{re} \left(e^{2i\varphi(x_l)} \int_{x_l}^{x_{l+1}} dy e^{2i\xi} v \right)$$

$$- \frac{1}{8\pi^2 l^2} \mathrm{im} \left[e^{4i\varphi(x_l)} \left(\int_{x_l}^{x} dy e^{2i\xi(y)} v(y) \right)^2 \right] - \frac{1}{4\pi^2 l^2} \mathrm{im}\, I_l(E) + O(l^{-1-\gamma}), \tag{2.10}$$

where

$$I_l(E) = \int_{x_l}^{x_{l+1}} dy \int_{y}^{x_{l+1}} ds e^{2i(\xi(s)-\xi(y))} v(s) v(y). \tag{2.11}$$

The basic properties of $I_l(E)$ are described by the following lemma.

Lemma 2.2. *For $v \in L_1(\mathbb{T})$, $I_l(E)$ admits a representation of the form*

$$I_l(E) = \mathcal{I}_l(E) + O(l^{1-\gamma}), \quad l \to \infty,$$

where $\mathcal{I}_l(E)$ is a C^1 function of E, satisfying the estimates

$$\mathcal{I}_l(E) = O(l), \quad \frac{d}{dE} \mathcal{I}_l(E) = O(l).$$

The proof of this lemma is given in appendix 1.

To derive a closed system for $R(x_l)$, $\theta(x_l)$ we need the following refinement of lemma 2.1.

Lemma 2.3. *Let $v \in L_1(\mathbb{T})$ and satisfy*

$$|\hat{v}^{(k)}(n)| \le C < n >^{-k}, \quad k = 1, 2, 3, \tag{2.12}$$

where

$$\hat{v}^{(k)}(n) = \hat{v}^{(k-1)}(n) - \hat{v}^{(k-1)}(n-1), \quad \hat{v}^{(0)} = \hat{v}.$$

Then,

$$\int\limits_{x_l}^{x_{l+1}} dy e^{2i\xi(y)} v(y) = e^{2i\omega_l} \pi \left(\frac{2l}{F}\right)^{1/2} \hat{v}(l) + t_{l+1} - t_l + O(l^{-\gamma}),$$

where

$$\omega_l = -\frac{\pi^3 l^3}{3F} + \pi l(\kappa' \ln l + E') + \frac{\pi}{8},$$

$$\kappa' = \frac{2\kappa}{F}, \quad E' = \frac{E - 2\kappa + \kappa \ln\left(\frac{\pi^2}{F}\right)}{F}.$$

$\{t_l\}$ is a bounded sequence: $|t_l| \le C$.

See appendix 1 for the proof.

Representations (2.9), (2.10) together with above lemma give:

$$\ln \frac{R(x_{l+1})}{R(x_l)} = \frac{1}{\sqrt{2lF}} \operatorname{im}\left(e^{2i\omega_l + 2i\varphi(x_l)} \hat{v}(l)\right) + \frac{1}{4lF} \operatorname{re}\left(e^{4i\omega_l + 4i\varphi(x_l)} \hat{v}^2(l)\right)$$

$$+\frac{1}{4lF}|\hat{v}(l)|^2 + \frac{1}{2\pi} \operatorname{im}\left(e^{2i\varphi(x_{l+1})} \frac{t_{l+1}}{l+1} - e^{2i\varphi(x_l)} \frac{t_l}{l}\right) + O(l^{-1-\gamma}), \tag{2.13}$$

$$\varphi(x_{l+1}) - \varphi(x_l) = \frac{1}{\sqrt{2lF}} \operatorname{re}\left(e^{2i\omega_l + 2i\varphi(x_l)} \hat{v}(l)\right) - \frac{1}{4lF} \operatorname{im}\left(e^{4i\omega_l + 4i\varphi(x_l)} \hat{v}^2(l)\right)$$

$$-\frac{1}{4\pi^2 l^2} \operatorname{im} \mathcal{I}_l(E) + \frac{1}{2\pi} \operatorname{re}\left(e^{2i\varphi(x_{l+1})} \frac{t_{l+1}}{l+1} - e^{2i\varphi(x_l)} \frac{t_l}{l}\right) + O(l^{-1-\gamma}). \tag{2.14}$$

Assume now that $\frac{F}{\pi^2} \in \mathbb{Q}$:

$$\frac{\pi^2}{F} = \frac{3p}{q}, \quad p, q \in \mathbb{N}.$$

We will consider $R(x_l)$, $\varphi(x_l)$ along the subsequence $l = qk$. Set

$$\tilde{R}(k) = R(x_{kq}), \quad \tilde{\varphi}(k) = \varphi(x_{kq}).$$

Then $\tilde{R}$, $\tilde{\varphi}$ solve the system:

$$\ln \frac{\tilde{R}(k+1)}{\tilde{R}(k)} = \operatorname{im}\left(e^{2i\Omega(k) + 2i\tilde{\varphi}(k)} b(k)\right) + \frac{1}{2} \operatorname{re}\left(e^{4i\Omega(k) + 4i\tilde{\varphi}(k)} b^2(k)\right)$$

$$+\frac{1}{2}|b(k)|^2 + \frac{1}{2\pi} \operatorname{im}\left(e^{2i\tilde{\varphi}(k+1)} \frac{t_{q(k+1)}}{k+1} - e^{2i\tilde{\varphi}(k)} \frac{t_{qk}}{k}\right) + O(k^{-1-\gamma}), \tag{2.15}$$

$$\tilde\varphi(k+1)-\tilde\varphi(k)=\operatorname{re}\left(e^{2i\Omega(k)+2i\tilde\varphi(k)}b(k)\right)-\frac{1}{2}\operatorname{im}\left(e^{4i\Omega(k)+4i\tilde\varphi(k)}b^2(k)\right)$$
$$-b_1(k)-\operatorname{im}\tilde{\mathcal{I}}_k(E)$$
$$+\frac{1}{2\pi}\operatorname{re}\left(e^{2i\tilde\varphi(k+1)}\frac{t_{q(k+1)}}{k+1}-e^{2i\tilde\varphi(k)}\frac{t_{qk}}{k}\right)+O(k^{-1-\gamma}). \tag{2.16}$$

Here

$$\Omega(k)=\pi kq(E'+\kappa'\ln(kq)),$$
$$b(k)=\frac{1}{\sqrt{2Fqk}}\hat v(qk)w(s(k)),\quad b_1(k)=\frac{1}{2Fqk}|\hat v(kq)|^2\operatorname{im}w_1(s(k)),$$
$$s(k)=\frac{d}{dk}\Omega(k)=\pi q(E'+\kappa'+\kappa'\ln(kq)),$$
$$w(s)=e^{i\pi/4}\sum_{r=0}^{q-1}e^{-2\pi i\frac{p}{q}r^3+2i\frac{s}{q}r},$$
$$w_1(s)=\sum_{r=1}^{q-1}e^{-2\pi i\frac{p}{q}r^3+2i\frac{s}{q}r}\sum_{r_1=0}^{r-1}e^{2\pi i\frac{p}{q}r_1^3-2i\frac{s}{q}r_1},\quad\text{if } q>1,$$

and $w_1(k)\equiv 0$ if $q=1$,

$$\tilde{\mathcal{I}}_k(E)=\frac{1}{4\pi^2q^2k^2}\sum_{l=kq}^{(k+1)q-1}\mathcal{I}_l(E).$$

2.1.3. Case of $\kappa=0$.

In this case (2.15) gives for $K_2>K_1$ sufficiently large

$$\ln\frac{\tilde R(K_2)}{\tilde R(K_1)}=\operatorname{im}\left(\sum_{k=K_1}^{K_2-1}e^{2i\pi E'qk+2i\tilde\varphi(k)}b(k)\right)+\frac{1}{2}\operatorname{re}\left(\sum_{k=K_1}^{K_2-1}e^{4i\pi E'qk+4i\tilde\varphi(k)}b^2(k)\right)$$
$$+\frac{1}{2}\sum_{k=K_1}^{K_2-1}|b(k)|^2+O(K_1^{-\gamma}). \tag{2.17}$$

Consider the second sum

$$\sum_{k=K_1}^{K_2-1}e^{4i\pi E'qk+4i\tilde\varphi(k)}b^2(k) \tag{2.18}$$

Summing by parts, one gets

$$(2.18)=\frac{1}{e^{-4\pi iE'q}-1}\sum_{k=K_1}^{K_2-1}e^{4i\pi E'qk}\left[e^{4i\tilde\varphi(k+1)}b^2(k+1)-e^{4i\tilde\varphi(k)}b^2(k)\right]$$
$$+O(K_1^{-1/2})=O(K_1^{-1/2}), \tag{2.19}$$

provided $2E'q\notin\mathbb{Z}$

In a similar way, one has

$$\sum_{k=K_1}^{K_2-1} e^{2i\pi E'qk+2i\tilde{\varphi}(k)}b(k)$$

$$= 2i\frac{1}{e^{-2\pi iE'q}-1}\sum_{k=K_1}^{K_2-1} e^{2i\pi E'qk+2i\tilde{\varphi}(k)}b(k)\,\mathrm{re}\left(e^{2i\pi E'qk+2i\tilde{\varphi}(k)}b(k)\right)+O(K_1^{-1/2})$$

$$= i\frac{1}{e^{-2\pi iE'q}-1}\sum_{k=K_1}^{K_2-1}|b(k)|^2+O(K_1^{-1/2}).$$

In particular,

$$\mathrm{im}\left(\sum_{k=K_1}^{K_2-1} e^{2i\pi E'qk+2i\tilde{\varphi}(k)}b(k)\right)=-\frac{1}{2}\sum_{k=K_1}^{K_2-1}|b(k)|^2+O(K_1^{-1/2}). \tag{2.20}$$

Combining (2.17), (2.19), (2.20),

$$\ln\frac{\tilde{R}(K_2)}{\tilde{R}(K_1)}=O(K_1^{-\gamma}).$$

This means that there exists

$$\lim_{k\to+\infty}\tilde{R}(k)=R_\infty,\quad 0<R_\infty<\infty,$$

which together with (2.4), (2.7) allows to conclude that any solution ψ of (2.1) satisfies for sufficiently large N

$$C_1(\psi)N^{1/2}\le\int_0^N dx|\psi|^2\le C_2(\psi)N^{1/2},$$

with some constants $C_1(\psi)$, $C_2(\psi)$ depending on ψ. Therefore, for $2E'q\notin\mathbb{Z}$ all solutions have the same rate of L_2 norm growth as $N\to+\infty$, and there is no subordinate solution on the right. By Gilbert-Pearson theory [12, 13], this implies that the singular continous spectrum of H is empty, the point spectrum is contained in the set $\{E: 2E'q\in\mathbb{Z}\}$ and

$$\Sigma_{ac}(H)=\mathbb{R}.$$

Notice also that for $\kappa=0$, H is unitary equivalent to $H+F$.

2.2. Case $\kappa\neq 0$: reduction to a model system.

2.2.1. Adiabatic regime.

Since $s(k)$, $b(k)$, $b_1(k)$ are slowly varying functions of k equations (2.15), (2.16) can be treated adiabatically except for relatively small vicinities of the stationary points K_m defined by the equation

$$\Omega'(K_m)=s(K_m)=\pi m,\quad m\in\mathbb{Z}$$

which means

$$K_m = A\Lambda^m, \quad \Lambda = e^{\frac{1}{q\kappa'}}, \quad A = \frac{1}{q} e^{-\frac{E'+\kappa'}{\kappa'}}. \tag{2.21}$$

For the sake of definiteness we will assume that $\kappa > 0$. So, it is the limit $m \to +\infty$ that we will be interested in.

We are going to study $\tilde{R}(k)$, $\tilde{\varphi}(k)$ along the subsequence k_m, $k_m = [\tilde{k}_m]$, where $\tilde{k}_m$ solves the equation

$$s(\tilde{k}_m) = \pi(m - \frac{1}{2}).$$

Define

$$\hat{K}_m = [K_m], \quad K_m^{\pm} = [K_m \pm K_m^{1-\eta}],$$

where $0 < \eta < \frac{1}{2}$ to be fixed later.

First we consider the intervals $J_m^{\pm}$, $J_m^- = [k_m, K_m^-]$, $J_m^+ = [K_m^+, k_{m+1}]$. Clearly, for $k \in J_m^- \cup J_m^+$,

$$\Omega^{(1)}(k) \equiv \Omega(k) - \Omega(k-1) = s(k) + O(k^{-1})$$

satisfies

$$|1 - e^{-2i\Omega^{(1)}(k)}| \geq C K_m^{-\eta},$$

provided m is sufficiently large.

Let us rewrite (2.15), (2.16) in the form

$$\ln \frac{\tilde{R}(K_2+1)}{\tilde{R}(K_1)} = \operatorname{im}\left(\sum_{k=K_1}^{K_2} e^{2i\Omega(k)+2i\tilde{\varphi}(k)} b(k)\right) + \frac{1}{2}\operatorname{re}\left(\sum_{k=K_1}^{K_2} e^{4i\Omega(k)+4i\tilde{\varphi}(k)} b^2(k)\right)$$

$$+ \sum_{k=K_1}^{K_2} \frac{1}{2}|b(k)|^2 + O(K_m^{-\gamma}), \tag{2.22}$$

$$\tilde{\varphi}(K_2+1) - \tilde{\varphi}(K_1) = \operatorname{re}\left(\sum_{k=K_1}^{K_2} e^{2i\Omega(k)+2i\tilde{\varphi}(k)} b(k)\right) - \frac{1}{2}\operatorname{im}\left(\sum_{k=K_1}^{K_2} e^{4i\Omega(k)+4i\tilde{\varphi}(k)} b^2(k)\right)$$

$$- \sum_{k=K_1}^{K_2} b_1(k) - \sum_{k=K_1}^{K_2-1} \operatorname{im} \tilde{\mathcal{I}}_k(E) + O(K_m^{-\gamma}), \tag{2.23}$$

$[K_1, K_2] \subset J_m^- \cup J_m^+$. Consider the sum

$$\sum_{k=K_1}^{K_2} e^{2i\Omega(k)+2i\tilde{\varphi}(k)} b(k) \tag{2.24}$$

Summing by parts one gets

$$(2.24) = \sum_{k=K_1}^{K_2} e^{2i\Omega(k)} \Big[(e^{-2i\Omega^{(1)}(k+1)} - 1)^{-1} e^{2i\tilde{\varphi}(k+1)} b(k+1)$$

$$- (e^{-2i\Omega^{(1)}(k)} - 1)^{-1} e^{2i\tilde{\varphi}(k)} b(k) \Big] + O(K_m^{-1/2+\eta})$$

$$= i \sum_{k=K_1}^{K_2} |b(k)|^2 (e^{-2is(k)} - 1)^{-1} + i \sum_{k=K_1}^{K_2} e^{4i\Omega(k)+4i\tilde{\varphi}(k)} b^2(k)(e^{-2is(k)} - 1)^{-1}$$

$$+ \sum_{k=K_1}^{K_2} e^{2i\Omega(k)+2i\tilde{\varphi}(k)} b(k) \left((e^{-2is(k+1)} - 1)^{-1} - (e^{-2is(k)} - 1)^{-1} \right) + O(K_m^{-1/2+\eta}). \tag{2.25}$$

The last sum in (2.25) can be estimated as folows.

$$\left| \sum_{k=K_1}^{K_2} e^{2i\Omega(k)+2i\tilde{\varphi}(k)} b(k) \left((e^{-2is(k+1)} - 1)^{-1} - (e^{-2is(k)} - 1)^{-1} \right) \right|$$

$$\le C \sum_{k=K_1}^{K_2} k^{-3/2} |e^{-2is(k)} - 1|^{-2} \le C \left(\int_{K_1}^{K_2} dy y^{-3/2} |e^{-2is(y)} - 1|^{-2} + K_m^{-3/2+3\eta} \right)$$

$$\le C K_m^{-1/2+\eta}. \tag{2.26}$$

Next we consider the sums

$$\sum_{k=K_1}^{K_2} e^{4i\Omega(k)+4i\tilde{\varphi}(k)} f(k), \tag{2.27}$$

where $k_m \le K_1 \le K_2 \le k_{m+1}$, and $f(k)$ is either $b^2(k)(e^{-2i\Omega^{(1)}(k-1)} - 1)^{-1}$ or $b^2(k)$. If K_1, K_2 satisfy $k_m + K_m^{1-\eta} \le K_1 \le K_2 \le K_m^-$ or $K_m^+ \le K_1 \le K_2 \le k_{m+1} - K_m^{1-\eta}$ then

$$|1 - e^{-4i\Omega^{(1)}(k)}| \ge C K_m^{-\eta}.$$

Proceeding in the same way as in (2.25) one gets

$$|(2.27)| \le C \left(\sum_{k=K_1}^{K_2} \left(k^{-3/2} ||e^{-2is(k)} - 1|^{-1} |e^{-4is(k)} - 1|^{-1} \right.\right.$$

$$\left.\left. + \; k^{-2} |e^{-2is(k)} - 1|^{-3} + k^{-2} |e^{-4is(k)} - 1|^{-3} \right) + K_m^{-1+2\eta} \right)$$

$$\le C \left(\int_{K_1}^{K_2} dy y^{-2} \left(|e^{-2is(y)} - 1|^{-3} + |e^{-4is(y)} - 1|^{-3} \right) + K_m^{-1/2+\eta} \right)$$

$$\le C K_m^{-1/2+\eta}.$$

On the other hand if $k_m \le K_1 \le K_2 \le k_m + K_m^{1-\eta}$, or $k_{m+1} - K_m^{1-\eta} \le K_1 \le K_2 \le k_{m+1}$, then

$$\sum_{k=K_1}^{K_2} e^{4i\Omega(k)+4i\tilde{\varphi}(k)} f(k) = O(K_m^{-\eta}).$$

Therefore, one has

$$\sum^{K_2} e^{4i\Omega(k)+4i\tilde{\varphi}(k)} f(k) = O(K_m^{-\gamma}), \tag{2.28}$$

for any K_1, K_2 such that $[K_1,\, K_2] \subset \mathcal{I}_m^- \cup \mathcal{I}_m^+$.

Combining (2.23), (2.24), (2.26) we get

$$\sum_{k=K_1}^{K_2} e^{2i\Omega(k)+2i\tilde{\varphi}(k)} b(k) = i \sum_{k=K_1}^{K_2} |b(k)|^2 (e^{-2is(k)} - 1)^{-1} + O(K_m^{-\gamma}). \tag{2.29}$$

Representations (2.15), (2.16), (2.26), (2.27) allow to conclude that for any K_1, K_2, $[K_1,\, K_2] \subset J_m^- \cup J_m^+$

$$\ln \frac{\tilde{R}(K_2+1)}{\tilde{R}(K_1)} = O(K_m^{-\gamma}), \tag{2.30}$$

$$\tilde{\varphi}(K_2+1) - \tilde{\varphi}(K_1) = -\frac{1}{2} \sum_{k=K_1}^{K_2} |b(k)|^2 \cot s(k) - \sum_{k=K_1}^{K_2} (b_1(k) + \operatorname{im} \tilde{I}_k(E)) + O(K_m^{-\gamma}). \tag{2.31}$$

The sums in the r.h.s. of (2.31) allow some further simplifications. One has

$$\sum_{k=K_1}^{K_2} |b(k)|^2 \cot s(k)$$

$$= \frac{|v_0|^2}{2Fq} \int_{K_1}^{K_2} dy y^{-1} |w(s(y)|^2 (\ln(qy))^{-2\beta} \cot s(y) + O(K_m^{-\gamma})$$

$$= |b_0|^2 \int_{s(K_1)}^{s(K_2)} ds |w(s)|^2 (s-s_0)^{-2\beta} \cot s + O(K_m^{-\gamma}), \tag{2.32}$$

where

$$b_0 = \frac{v_0}{\sqrt{2Fq}} \mu^{-1/2+\beta}, \quad \mu = \pi q \kappa', \quad s_0 = \pi q (E' + \kappa').$$

In a similar way,

$$\sum_{k=K_1}^{K_2} b_1(k) = |b_0|^2 \int_{s(K_1)}^{s(K_2)} ds \operatorname{im} w_1(s) (s-s_0)^{-2\beta} + O(K_m^{-\gamma}). \tag{2.33}$$

(2.31), (2.32), (2.33) give

$$\tilde{\varphi}(K_m^-) - \tilde{\varphi}(k_m) = -\frac{|b_0|^2}{2} |w(\pi m)|^2 (\pi m - s_0)^{-2\beta} \ln(\mu K_m^{-\eta})$$
$$-\frac{|b_0|^2}{2} \int_{\pi(m-1/2)}^{\pi m} ds \cot s \big(|w(s)|^2 (s-s_0)^{-2\beta} - |w(\pi m)|^2 (\pi m - s_0)^{-2\beta} \big)$$
$$-|b_0|^2 \int_{\pi(m-1/2)}^{\pi m} ds \operatorname{im} w_1(s) (s-s_0)^{-2\beta} - \sum_{k=k_m}^{K_m^-} \operatorname{im} \tilde{I}_k(E) + O(K_m^{-\gamma}), \tag{2.34}$$

$$\tilde{\varphi}(k_{m+1}) - \tilde{\varphi}(K_m^+) = \frac{|b_0|^2}{2} |w(\pi m)|^2 (\pi m - s_0)^{-2\beta} \ln(\mu K_m^{-\eta})$$
$$-\frac{|b_0|^2}{2} \int_{\pi m + \mu K_m^{-\eta}}^{\pi(m+1/2)} ds \cot s \big(|w(s)|^2 (s-s_0)^{-2\beta} - |w(\pi m)|^2 (\pi m - s_0)^{-2\beta} \big)$$
$$-|b_0|^2 \int_{\pi m}^{\pi(m+1/2)} ds \operatorname{im} w_1(s) (s-s_0)^{-2\beta} - \sum_{k=K_m^+}^{k=k_{m+1}} \operatorname{im} \tilde{I}_k(E) + O(K_m^{-\gamma}). \tag{2.35}$$

2.2.2. Vicinities of stationary points. In this subsection we analyse system (2.15), (2.16) in the $K_m^{1-\eta}$- vicinity of the turning point K_m:

$$k \in \delta_m, \quad \delta_m = [K_m^-, K_m^+].$$

In this vicinity one has

$$b(k)e^{2i\Omega(k)} = \left(\frac{\mu}{K_m}\right)^{1/2} d_m e^{2i\Phi_0(m)+i\mu\frac{(k-K_m)^2}{K_m}} + O(K_m^{1-3\eta}),$$

$$d_m = b_0 w(\pi m)(\pi m - s_0)^{-\beta}, \quad \Phi_0(m) = \Omega(K_m) - \pi m K_m, \tag{2.36}$$

provided $\frac{1}{3} < \eta < \frac{1}{2}$. By (2.21),

$$\Phi_0(m) = \rho \Lambda^m,$$

where

$$\rho = -\pi\kappa' e^{-\frac{E'+\kappa'}{\kappa'}},$$

one can consider ρ as a new spectral parameter.

From now on we fix η in such a way that

$$\frac{3}{8} < \eta < \frac{1}{2}. \tag{2.37}$$

Introduce the vectors $\chi(k) \in \mathbb{C}^2$,

$$\chi(k) = \tilde{R}(k)e^{i\tilde{\varphi}(k)\sigma_3}\begin{pmatrix}1\\1\end{pmatrix}, \quad \sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

Notice that

$$\chi(k) = e^{-i\xi(x_{qk})\sigma_3}\left(Q_\xi(\xi(x_{qk}))\begin{pmatrix}1\\1\end{pmatrix} + Q(\xi(x_{qk}))\begin{pmatrix}1\\-1\end{pmatrix}\right).$$

For $k \in \delta_m$, (2.15), (2.16) imply

$$\chi(k+1) = \chi(k) + i\begin{pmatrix}\bar{b}(k)e^{-2i\Omega(k)-2i\tilde{\varphi}(k)} & 0\\ 0 & b(k)e^{2i\Omega(k)+2i\tilde{\varphi}(k)}\end{pmatrix}\chi(k) + O(k^{-1})\chi(k)$$

$$= (A_0(k,m) + A_1(k,m,\chi))\chi(k), \tag{2.38}$$

where

$$A_0 = I + \left(\frac{\mu}{K_m}\right)^{1/2} D_m e^{i\mu\frac{(k-K_m)^2}{K_m}\sigma_3}, \quad D_m = i\begin{pmatrix}0 & \bar{d}_m e^{-2i\Phi_0(m)}\\ -d_m e^{2i\Phi_0(m)} & 0\end{pmatrix},$$

and

$$A_1 = O(K_m^{1-3\eta}). \tag{2.39}$$

On the interval δ_m system (2.38) can be approximated by the differential equation

$$\frac{d}{dk}\chi = \left(\frac{\mu}{K_m}\right)^{1/2} D_m e^{i\mu\frac{(k-K_m)^2}{K_m}\sigma_3}\chi. \tag{2.40}$$

Set

$$\chi(k) = \psi(y), \quad y = \left(\frac{\mu}{K_m}\right)^{1/2}(k - K_m).$$

Then (2.40) takes the form

$$\frac{d}{dy}\psi = D_m e^{iy^2\sigma_3}\psi. \tag{2.41}$$

Note that if $\psi(y)$ is a solution then $\sigma_1\overline{\psi(y)}$, $\sigma_3\psi(y)$ also satisfy (2.41). Here $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Since d_m does not depend on y, (2.41) can be solved explicitely in terms of Hermite functions. One checks directly that

$$\psi(y) = \begin{pmatrix} H_{\lambda_m}(e^{-i\pi/4}y) \\ -d_m e^{2i\Phi_0(m)+iy^2-i\pi/4} H_{\lambda_m-1}(e^{-i\pi/4}y) \end{pmatrix}, \quad \lambda_m = -\frac{i}{2}|d_m|^2, \tag{2.42}$$

is a solution of (2.41). Here $H_\lambda(z)$ stands for the standard Hermite function: it satisfies the equation

$$f_{zz} - 2zf_z + 2\lambda f = 0,$$

and

$$H_\lambda(z) = (2z)^\lambda(1 + O(z^{-2})), \quad z \to \infty,$$

$-\pi/2 < \arg z \le \pi/2$.

Let us introduce the matrix solutions $\Psi^\pm(y)$:

$$\Psi^+(y) = (\psi(y), \sigma_1\overline{\psi(y)}), \quad \Psi^-(y) = \sigma_3\Psi^+(-y)\sigma_3. \tag{2.43}$$

As $y \to +\infty$,

$$\Psi^+(y) = e^{-\frac{\pi|d_m|^2}{8}}(2y)^{\lambda_m\sigma_3} + O(y^{-1}), \tag{2.44}$$

uniformely with respect to m sufficiently large.

The determinat of $\Psi^\pm(y)$ does not depend on y:

$$\det\Psi^\pm = e^{-\frac{\pi|d_m|^2}{4}}.$$

Using Ψ^- one can rewrite full equation (2.38) in the form

$$\chi(k) = \Psi^-(y)a_-(k), \quad y = \left(\frac{\mu}{K_m}\right)^{1/2}(k - K_m) \tag{2.45}$$

$$a_-(k) = a(K_m^-) + \sum_{j=K_m^-}^{k-1} A_2(j,m)a_-(j), \quad k \ge K_m^- + 1, \tag{2.46}$$

$$A_2(k,m) = \left(\Psi^-\left(y + \left(\frac{\mu}{K_m}\right)^{1/2}\right)\right)^{-1}\left[\Psi^-(y) + \left(\frac{\mu}{K_m}\right)^{1/2}\frac{d}{dy}\Psi^-(y)\right.$$
$$\left. -\Psi^-\left(y + \left(\frac{\mu}{K_m}\right)^{1/2}\right) + A_1(k,m)\Psi^-(y)\right].$$

It folows directly from (2.40), (2.41) that

$$|\Psi^{\pm}| \le C, \quad \left|\frac{d\Psi^{\pm}}{dy}\right| \le C, \quad \left|\frac{d^2\Psi^{\pm}}{dy^2}\right| \le C < y >, \quad y \in \mathbb{R}, \tag{2.47}$$

which together with (2.39) implies

$$|A_2| \le CK_m^{1-3\eta}, \quad k \in \delta_m. \tag{2.48}$$

(2.46), (2.48) allow to conclude

$$|a_-(k) - a_-(K_m^-)| \le CK_m^{-\gamma} \max_{k\in\delta_m} |a_-(k)|, \quad k \in \delta_m, \tag{2.49}$$

provided (2.37) is satisfied. This means that for m sufficiently large

$$|a_-(k) - a_-(K_m^-)| \le CK_m^{-\gamma} a_-(K_m^-), \quad k \in \delta_m, \tag{2.50}$$

In a similar way, setting

$$\chi(k) = \Psi^+(y) a_+(k), \tag{2.51}$$

one gets

$$|a_+(k) - a_-(K_m^+)| \le CK_m^{-\gamma} |a_-(K_m^+)|, \quad k \in \delta_m. \tag{2.52}$$

Comparing (2.46), (2.51) and taking into account (2.44), (2.47), (2.50), (2.52) we obtain

$$\chi(K_m^+) = (2\sqrt{\mu}K_m^{1/2-\eta})^{\lambda_m\sigma_3} \Psi_+^{-1}(0)\sigma_3\Psi_+(0)\sigma_3 (2\sqrt{\mu}K_m^{1/2-\eta})^{-\lambda_m\sigma_3}\chi(K_m^-)$$

$$+O(K_m^{-\gamma})\tilde{R}(K_m^-). \tag{2.52}$$

By (2.42) the expression $\Psi_+^{-1}(0)\sigma_3\Psi_+(0)\sigma_3$ may be represented as

$$\Psi_+^{-1}(0)\sigma_3\Psi_+(0)\sigma_3 = e^{-i\Phi_0(m)\sigma_3} S(d_m) e^{i\Phi_0(m)\sigma_3},$$

$$S(\xi) = S_0^{-1}(\xi)\sigma_3 S_0(\xi)\sigma_3,$$

$$S_0(\xi) = \begin{pmatrix} H_\lambda(0) & -e^{i\pi/4}\bar{\xi} H_{-\lambda-1}(0) \\ -e^{-i\pi/4}\xi H_{\lambda-1}(0) & H_{-\lambda}(0) \end{pmatrix}, \quad \lambda = -\frac{i}{2}|\xi|^2. \tag{2.53}$$

Clearly,

$$S(\xi) = \begin{pmatrix} s(\xi) & \bar{r}(\xi) \\ r(\xi) & s(\xi) \end{pmatrix}, \quad s^2(\xi) - |r(\xi)|^2 = 1,$$

$$r(\xi) = 2e^{-i\pi/4+\pi|\xi|^2/4} H_\lambda(0) H_{\lambda-1}(0), \quad s(\xi) = e^{\pi|\xi|^2/4}(|H_\lambda(0)|^2 + |\xi|^2 |H_{\lambda-1}(0)|^2).$$

As a consequence,

$$r(d_m) = r_0 w(\pi m) m^{-\beta}(1 + O(m^{-2\beta})), \quad s(d_m) = 1 + O(m^{-2\beta}), \quad r_0 = e^{-i\pi/4}\sqrt{\pi} b_0,$$

$$\frac{\partial}{\partial \ldots} r(d_m) = O(m^{-1-2\beta}).$$

(2.52) together with (2.30), (2.34), (2.35) leads to a “closed system” for $\hat{\chi}(m) = \chi(k_m)$:

$$\hat{\chi}(m+1) = \mathcal{A}(m)\hat{\chi}(m) + O(e^{-m\gamma}|\hat{\chi}(m)|), \tag{2.54}$$

$$\mathcal{A}(m) = e^{-i(\Phi_0(m)+\triangle_+(m))\sigma_3} S(d_m) e^{-i(\Phi_0(m)+\triangle_-(m))\sigma_3},$$

where

$$\triangle_\pm(m) = \Phi_1(m,\rho) + \Phi_2^\pm(m,\rho) + \Phi_3^\pm(m,\rho),$$

$$\Phi_1(m,\rho) = -\frac{|d_m|^2}{4}\ln\left(\frac{\mu}{4K_m}\right),$$

$$\Phi_2^-(m,\rho) = -\frac{|b_0|^2}{2}\int_{\pi(m-1/2)}^{\pi m} ds\cot s\big(|w(s)|^2(s-s_0)^{-2\beta} - |w(\pi m)|^2(\pi m - s_0)^{-2\beta}\big)$$

$$-|b_0|^2\int_{\pi(m-1/2)}^{\pi m} ds\,\mathrm{im}\, w_1(s)(s-s_0)^{-2\beta},$$

$$\Phi_2^+(m,\rho) = \frac{|b_0|^2}{2}\int_{\pi m}^{\pi(m+1/2)} ds\cot s\big(|w(s)|^2(s-s_0)^{-2\beta} - |w(\pi m)|^2(\pi m - s_0)^{-2\beta}\big)$$

$$+|b_0|^2\int_{\pi m}^{\pi(m+1/2)} ds\,\mathrm{im}\, w_1(s)(s-s_0)^{-2\beta}$$

$$\Phi_3^-(m,\rho) = -\sum_{k=k_m}^{\hat{K}_m} \mathrm{im}\,\tilde{I}_k(E),$$

$$\Phi_3^+(m,\rho) = \sum_{k=\hat{K}_m}^{k=k_{m+1}} \mathrm{im}\,\tilde{I}_k(E).$$

Note that $\Phi_1, \Phi_2^\pm$ are smooth functions of ρ satisfying

$$|\Phi_1(m,\rho)| \le Cm^{1-2\beta}, \quad \left|\frac{\partial}{\partial\rho}\Phi_1(m,\rho)\right| \le Cm^{-2\beta}, \tag{2.55}$$

$$|\Phi_2^\pm(m,\rho)| \le Cm^{-2\beta}, \quad \left|\frac{\partial}{\partial\rho}\Phi_2^\pm(m,\rho)\right| \le Cm^{-1-2\beta}. \tag{2.56}$$

It follows from lemma 2.2 that

$$|\Phi_3^\pm(m,\rho)| \le C \tag{2.57}$$

$$\big|\Phi_3^\pm(m,\rho) - \Phi_3^\pm(m;\rho')\big| \le C(\Lambda^{-m} + |\rho-\rho'|). \tag{2.58}$$

Fix a function $\varphi \in C_0^\infty(\mathbb{R})$ such that $\varphi \ge 0$, $\int_{\mathbb{R}} dx\varphi(x) = 1$. Set

$$\tilde{\Phi}_3^\pm(m,\rho) = \kappa_m\int_{\mathbb{R}} dy\varphi(\kappa_m(y-\rho))\Phi_3^\pm(m,y),$$

where $\kappa_m = \Lambda_1^{-m}$, $1 < \Lambda_1 < \Lambda$ to be fixed later. Then (2.58) implies

$$|\Phi_3^\pm(m,\rho) - \tilde{\Phi}_3^\pm(m,\rho)| \le Ce^{-\gamma m}$$

$$\left|\frac{\partial}{\partial\rho}\tilde{\Phi}_3^{\pm}(m,\rho)\right| \leq C\Lambda_1^m. \tag{2.59}$$

Therefore, one can rewrite (2.54) in the form

$$\hat{\chi}(m+1) = \mathcal{A}_0(m)\hat{\chi}(m) + O(e^{-m\gamma}|\hat{\chi}(m)|), \tag{2.60}$$

$$\mathcal{A}_0(m) = e^{-i\Gamma_+(m)\sigma_3}S(d_m)e^{i\Gamma_-(m)\sigma_3},$$

where

$$\Gamma_\pm(m) = \rho\Lambda^m + \Gamma_1^\pm(m), \quad \Gamma_1^\pm(m) = \Phi_1(m,\rho) + \Phi_2^\pm(m,\rho) + \tilde{\Phi}_3^\pm(m,\rho).$$

As a simple consequence of (2.60)(=(2.54)) one obtains

Proposition 2.1. *The operator H has no point spectrum.*

Proof. (2.60) implies

$$\tilde{R}(k_{m+1}) \geq (1 - Cm^{-\beta})\tilde{R}(k_m).$$

As a consequence, one gets

$$\tilde{R}(k_m) \geq C(\psi)e^{-Cm^{1-\beta}}.$$

Combining this estimate with (2.51), (2.52), (2.30), (2.13), (2.7), (2.4) one can check easily that for any solution ψ

$$\int_0^N dx|\psi|^2 \geq C(\psi)N^{1/2}e^{-C(\ln N)^{1-\beta}}.$$

Therefore, ψ can not belong to L_2. ∎

3. Analysis of the model system

In this section we study the model system

$$\chi(m+1) = \mathcal{A}_0(m)\chi(m). \tag{3.1}$$

3.1. Positivity of the Lyapounov exponent.

3.1.1. Prüfer coordinates for (3.1). We denote by $\chi_\alpha(m)$ the solution of (3.1) satisfying

$$\chi_\alpha(M_0) = e^{i\alpha\sigma_3}\begin{pmatrix}1\\1\end{pmatrix}, \quad \alpha \in [0,\pi),$$

M_0 is supposed to be a large fixed number.

Define the Prüfer variables $R_\alpha(m)$, $\varphi_\alpha(m)$ by

$$\chi_\alpha(m) = R_\alpha(m)e^{i\varphi_\alpha(m)\sigma_3}\begin{pmatrix}1\\1\end{pmatrix}.$$

They solve

$$\frac{R_\alpha(m+1)}{R_\alpha(m)} = |s(d_m) + r(d_m)e^{2i\Gamma_-(m)}\zeta(m)|, \tag{3.2}$$

$$\frac{\zeta(m+1)}{\zeta(m)} = e^{2i\triangle\Gamma(m)} \frac{s(d_m) + \bar{r}(d_m)\bar{\zeta}(m)e^{-2i\Gamma_-(m)}}{s(d_m) + r(d_m)\zeta(m)e^{2i\Gamma_-(m)}}, \tag{3.3}$$

where

$$\triangle\Gamma(m) = \Gamma_-(m) - \Gamma_+(m), \quad \zeta(m) = e^{2i\varphi_\alpha(m)}.$$

From (3.2) we have

$$\ln R_\alpha(M) = \frac{1}{2}|r_0|^2 n(M)(1 + O(n^{-\gamma}(M)))$$

$$+\operatorname{re}\left(r_0 \sum_{m=M_0}^{M} m^{-\beta} w(\pi m) e^{2i\Gamma_-(m)} \zeta(m)\right) \tag{3.4}$$

$$-\frac{1}{2}\operatorname{re}\left(r_0^2 \sum_{m=M_0}^{M} m^{-2\beta} w^2(\pi m) e^{4i\Gamma_-(m)} \zeta^2(m)\right), \tag{3.5}$$

where

$$n(M) = \sum_{m=1}^{M} m^{-2\beta} |w(\pi m)|^2.$$

Note that

$$w(\pi m) = w(\pi(m+q)), \quad \sum_{j=0}^{q-1} |w(\pi j)|^2 = q^2.$$

As a consequence,

$$n(M) = q \sum_{m=M_0}^{M} m^{-2\beta} + O(1) = \frac{q}{1-2\beta} M^{1-2\beta} + O(1),$$

as $M \to +\infty$.

Notice also that (3.3) implies

$$|\zeta_\rho'(m+1)| \le |\zeta_\rho'(m)|(1 + Cm^{-\beta}) + Cm^{-\beta}\Lambda^m \le Cm^{-\beta}\Lambda^m. \tag{3.6}$$

Next two subsubsections are devoted to the proof of the following result.

Proposition 3.1. *For any α and for a.e. ρ we have*

$$\lim_{m\to+\infty} \frac{\ln R_\alpha(m)}{m^{1-2\beta}} = r_*,$$

where $r_* = \frac{|r_0|^2 q}{2(1-2\beta)}$.

3.1.2. Estimates of sum (3.5).

To prove proposition 3.1 we are going to analyse the sums

$$\Sigma_1(M) = \sum m^{-\beta} w(\pi m) e^{2i\Gamma_-(m)} \zeta(m), \tag{3.7}$$

$$\Sigma_2(M) = \sum_{m=M_0}^{M} m^{-2\beta} w^2(\pi m) e^{4i\Gamma_-(m)} \zeta^2(m) \tag{3.8}$$

and show that they are $o(n(M))$. In this part of the paper we follow closely the arguments of [6, 8, 14]. We start by a technical lemma, which is essentially lemma 10.1 of [6]. Consider the sums of the form

$$S_N(\rho) = \sum_{n=0}^{N} a(n) f_n(L^{h+n}\rho) g(L^{h+n}\rho),$$

where $N \geq 1$, $h \in \mathbb{R}$, $a(n)$ are real numbers, $\{f_n\} \in C^1$, $\|f_n\|_\infty \leq 1$, $g(y) = \cos(ky + b)$. We assume

$$\|f_n'\|_\infty \leq K,$$
$$|k| + |k|^{-1} \leq K, \quad L \geq \Lambda > 1.$$

Lemma 3.1. *One has*

$$\sup_{I, |I|=1} \int_I d\rho \exp(tS_N(\rho)) \leq e^{B(K,\Lambda)t^2 A^2(N) + B(K,\Lambda)t\mathcal{Q}(N)}, \tag{3.9}$$

$$A^2(N) = \sum_{n=0}^{N} a^2(n), \quad \mathcal{Q}(N) = \sum_{n=0}^{N} |a(n)|(\|f_n'\|_\infty + L^{-(h+n)}),$$

provided

$$0 \leq t \max_{0 \leq n \leq N} |a(n)| \leq 1,$$

the supremum being taken over all intervals $I \subset \mathbb{R}$, $|I| = 1$.

Here and below $B(K, \Lambda)$ are positive constants that depend only on K, Λ, they may change from line to line.

The proof of this lemma is given in appendix 2.

Applying (3.9) to sum (3.8) one gets the following result. Fix an interval $I \subset \mathbb{R}$, $|I| = 1$.

Lemma 3.2. *There exist $\varepsilon_0, \varepsilon_1 > 0$, such that*

$$\text{mes}\{\rho \in I : |\Sigma_2(M)| \geq M^{1-2\beta-\varepsilon_0}\} \leq e^{-M^{\varepsilon_1}}, \tag{3.10}$$

provided M is sufficiently large.

Proof. Let $e^{4i\Gamma_1^-(m)}\zeta^2(m) = \xi_m(\rho) + i\eta_m(\rho)$, $m^{-2\beta}w^2(\pi m) = a_r(m) + ia_i(m)$, $\xi_m, \eta_m, a_r(m), a_i(m) \in \mathbb{R}$. We will prove (3.10) with $\Sigma_2(M)$ replaced by the sum

$$\sum_{m=M_0}^{M} a_r(m)\xi_m(\rho)\cos(4\Lambda^m\rho), \tag{3.11}$$

the others cases being similar. We break sum (3.11) into two parts: $\sum_{M_0}^{M} = \sum_{M_0}^{M_1} + \sum_{M_1}^{M}$, where $1 \ll M_1 \ll M$ to be specified later. The first sum can be estimated trivially

$$\sum^{M_1} a_r(m)\xi_m(\rho)\cos(4\Lambda^m\rho) = O(M_1^{1-2\beta}). \tag{3.12}$$

Consider the second one. Let $L = \Lambda$, $h = M_1$, $f_n(\rho) = \xi_{M_1+n}(\Lambda^{-(M_1+n)}\rho)$. Clearly, $\|f_n\|_\infty \le 1$. By (2.59), (2.55), (2.56), (3.6),

$$\left|\frac{\partial}{\partial\rho} f_n(\rho)\right| \le C(M_1+n)^{-\beta}. \tag{3.13}$$

From (3.9), (3.13), one gets for any $\delta \ge 0$, $t \ge 0$, :

$$\operatorname{mes}\{\rho \in I : \sum_{m=M_1}^{M} a_r(m)\xi_m(\rho)\cos(4\Lambda^m\rho) \ge \delta\}$$

$$\le e^{-\delta t}\int_I d\rho \exp\left(t\sum_{m=M_1}^{M} a_r(m)\xi_m(\rho)\cos(4\Lambda^m\rho)\right)$$
$$\le e^{-\delta t}e^{Ct^2\sum_{M_1}^{M} m^{-4\beta}+Ct\sum_{M_1}^{M} m^{-3\beta}}, \tag{3.14}$$

provided $tM_1^{-2\beta}$ is suficiently small:

$$q^2 t M_1^{-2\beta} \le 1.$$

Choosing $t = c\frac{\delta}{\sum_{M_1}^{M} m^{-4\beta}}$ with a sufficiently small constant c one gets

$$\operatorname{mes}\{\rho \in I : |\sum_{m=M_1}^{M} a_r(m)\xi_m(\rho)\cos(4\Lambda^m\rho)| \ge \delta\}$$

$$\le 2\exp\left(-C\frac{\delta^2}{\sum_{M_1}^{M} m^{-4\beta}}\right), \tag{3.15}$$

provided

$$c_1^{-1}\sum_{m=M_1}^{M} m^{-3\beta} \le \delta \le c_1 M_1^{2\beta}\sum_{m=M_1}^{M} m^{-4\beta}, \tag{3.16}$$

for some suitable constant c_1. We consider the cases

$$1. \quad 0 < \beta < \frac{1}{3}$$

$$2. \quad \frac{1}{3} \le \beta < \frac{1}{2}.$$

In the first case we set

$$\delta = M^{1-2\beta-\varsigma_1}, \quad M_1 = M^{1-\varsigma_2}, \quad \varsigma_j > 0. \tag{3.17}$$

Choose ς_j in such a way that

$$0 < \varsigma_1 < \beta, \quad 0 < \varsigma_2 < \frac{\varsigma_1}{2\beta}. \tag{3.18}$$

This gives

$$\delta^{-1}M^{1-3\beta}, \delta M_1^{-2\beta}\left(\sum_{m=M_1}^{M} m^{-4\beta}\right)^{-1} = O(M^{-\gamma}),$$

which means that (3.16) is satisfied provided M is sufficiently large. Combining (3.12), (3.15), (3.17) one gets (3.10) with, say,

$$\varepsilon_0 < \frac{\beta}{2}, \quad \varepsilon_1 \leq 1 - 3\beta.$$

Consider the case $\frac{1}{3} \leq \beta < \frac{1}{2}$. Set

$$\delta = M_1^{\varsigma_3}, \quad M_1 = M^{\varsigma_4}, \quad 0 < \varsigma_4 < 1.$$

Then (3.16) is satisfied provided

$$1 - 3\beta < \varsigma_3 < 1 - 2\beta.$$

As a consequence, one obtains (3.10) for any ε_0, ε_1 such that

$$\varepsilon_0 < 1 - 2\beta, \quad \varepsilon_1 < 1 - 2\beta - \varepsilon_0.$$

■

Since the right hand sides of (3.10) belong to l_1, lemma 3.2 implies immediately

Lemma 3.3. *For any α and a.e. ρ,*

$$\lim_{m\to+\infty} \frac{\Sigma_2(m)}{m^{1-2\beta-\varepsilon}} = 0, \tag{3.19}$$

provided $\varepsilon < \varepsilon_0$.

3.1.3. Estimates of $\Sigma_1(M)$.

Consider sum (3.7). Iteration of (3.3) gives

$$\zeta(m+T) = e^{2i\sum_{s=1}^{T}\triangle\Gamma(m+T-s)}\zeta(m)$$

$$+\bar{r}_0\sum_{s=1}^{T}(m+T-s)^{-\beta}\overline{w(\pi(m+T-s))}e^{2i\sum_{k=1}^{s}\triangle\Gamma(m+T-k)-2i\Gamma_-(m+T-s)}$$

$$-r_0\sum_{s=1}^{T}(m+T-s)^{-\beta}w(\pi(m+T-s))e^{2i\sum_{k=1}^{s}\triangle\Gamma(m+T-k)+2i\Gamma_-(m+T-s)}\zeta^2(m+T-s)$$

$$+O(m^{-2\beta}T),$$

for any $T > 0$. Returning to (3.7) one gets for M sufficiently large and T, $1 \ll T \ll M$, to be specified below,

$$\Sigma_1(M) = \sum^{2T} m^{-\beta}w(\pi m)e^{2i\Gamma_-(m)}\zeta(m)$$

$$+\sum_{m=2T}^{M} m^{-\beta} w(\pi m) e^{2i\Gamma_-(m)+2i\sum\limits_{s=1}^{T}\triangle\Gamma(m-s)}\zeta(m-T) \tag{3.20}$$

$$+\bar{r}_0\sum_{s=1}^{T}\sum_{m=2T}^{M} m^{-2\beta} w(\pi m)\overline{w(\pi(m-s))} e^{2i\Gamma_-(m)-2i\Gamma_-(m-s)+2i\sum\limits_{k=1}^{T}\triangle\Gamma(m-k)} \tag{3.21}$$

$$-r_0\sum_{s=1}^{T}\sum_{m=2T}^{M} m^{-2\beta} w(\pi m) w(\pi(m-s)) e^{2i\Gamma_-(m)+2i\Gamma_-(m-s)+2i\sum\limits_{k=1}^{T}\triangle\Gamma(m-k)}\zeta^2(m-s) \tag{3.22}$$

$$+O(T\sum_{m=2T}^{M} m^{-3\beta}) + O(T^{2-2\beta}).$$

We choose T as follows:

$$T = \left[\frac{\ln M}{\ln \Lambda}\right] + 1.$$

Then for the first sum one has the trivial estimate

$$\left|\sum_{m=M_0}^{M_1} m^{-\beta} w(\pi m) e^{2i\Gamma_-(m)}\zeta(m)\right| \le CT^{1-\beta} \le C(\ln M)^{1-\beta}. \tag{3.23}$$

We next consider sum (3.20) and prove the followng estimate.

Lemma 3.4. *For any positive z satisfying $z < 1-2\beta$, one has*

$$\operatorname{mes}\{\rho \in I : |(3.20)| \ge M^{1-2\beta-z}\} \le Ce^{-CM^{1-2\beta-2z}}, \tag{3.24}$$

provided M is sufficiently large.

Proof. The proof is similar to that of lemma 3.2. Write $w(\pi m)m^{-\beta} = a_r^1(m) + a_i^1(m)$, $e^{2i\sum\limits_{s=1}^{T}\triangle\Gamma(m-s)}\zeta(m-T) = \xi_m^1(\rho) + \eta_m^1(\rho)$, $a_r^1(m), a_i^1(m), \xi^1(m), \eta^1(m) \in \mathbb{R}$. It is sufficient to prove (3.24) for the sum

$$\sum_{m=2T}^{M} a_r^1(m)\xi_m^1(\rho)\cos(2\Lambda^m\rho).$$

Clearly, one has,

$$|a^1(m)| \le Cm^{-\beta},\ |\xi_m^1(\rho)| \le 1,\ \left|\frac{\partial}{\partial\rho}\xi_m^1(\rho)\right| \le C(\Lambda^{m-T}m^{-\beta} + \Lambda_1^m).$$

Applying (3.9) one gets for $t, \delta \ge 0$

$$\operatorname{mes}\{\rho \in I : |\sum_{m=2T}^{M} a_r^1(m)\xi_m^1(\rho)\cos(2\Lambda^m\rho)| \ge \delta\}$$

$$\le 2\exp\left(-\delta t + Ct^2M^{1-2\beta} + Ct\Lambda^{-T}M^{1-2\beta} + Ct\left(\frac{\Lambda_1}{\Lambda}\right)^{2T}\right),$$

provided

$$CT^{-\beta}t \leq 1.$$

This gives

$$\mathrm{mes}\{\rho \in I : |\sum_{m=2T}^{M} a_r^1(m)\xi_m^1(\rho)\cos(2\Lambda^m\rho)| \geq \delta\}$$

$$\leq 2e^{-C\delta^2 M^{2\beta-1}},$$

provided δ satisfies

$$C(\Lambda^{-T}M^{1-2\beta} + \left(\frac{\Lambda_1}{\Lambda}\right)^{2T}) \leq \delta \leq CT^{\beta}M^{1-2\beta}.$$

Clearly, $\delta = M^{1-2\beta-z}$ verifies these conditions. As a consequence, one gets (3.24). ■

Notice that choosing $z < \frac{1-2\beta}{2}$ one makes the r.h.s. of (3.24) to be in l_1, which implies:

Lemma 3.5. *For a.e.* ρ

$$\lim_{m\to+\infty} \frac{(3.20)}{m^{1-2\beta-z}} = 0,$$

provided $z < \frac{1-2\beta}{2}$.

Consider sums (3.21), (3.22).

Lemma 3.6. (3.21), (3.22) *satisfy for some* $\varepsilon_0, \varepsilon_1 > 0$,

$$\mathrm{mes}\{\rho \in I : |(3.21)| + |(3.22)| \geq M^{1-2\beta-\varepsilon_0}\} \leq e^{-M^{\varepsilon_1}}, \tag{3.25}$$

provided M *is sufficiently large.*

Proof. Sums (3.21), (3.22) have the following structure

$$\sum_{s=1}^{T}\sum_{m=2T}^{M} t(m,s)\Psi(m,s)e^{2i\phi(m,s)},$$

where

$$t(m,s) = \bar{r}_0 m^{-2\beta} w(\pi m)\overline{w(\pi(m-s))}, \quad \Psi(m,s) = e^{2i\sum_{k=1}^{T}\triangle\Gamma(m-k)},$$

$$\phi(m,s) = \Gamma_-(m) - \Gamma_-(m-s),$$

in the case of (3.21), and

$$t(m,s) = -r_0 m^{-2\beta} w(\pi m) w(\pi(m-s)), \quad \Psi(m,s) = e^{2i\sum_{k=1}^{T}\triangle\Gamma(m-k)}\zeta^2(m-s),$$

$$\phi(m,s) = \Gamma_-(m) + \Gamma_-(m-s),$$

for (3.22),

As in the proof of lemma 3.2 we write $t(m,s) = a_r(m,s) + ia_i(m,s)$, $\Psi(m,s) = \xi(m,s) + i\eta(m,s)$, $a_r(m,s)$, $a_i(m,s)$, $\xi(m,s)$, $\eta(m,s) \in \mathbb{R}$ and prove (3.25) for the sum

$$\sum_{s=1}^{T} \sum_{m=2T}^{M} a_r(m,s)\xi(m,s)\cos(2\phi(m,s)). \tag{3.26}$$

Clearly,

$$|a_r(m,s)| \le Cm^{-2\beta}, \quad |\xi(m,s)| \le 1, \quad \left|\frac{\partial}{\partial\rho}\xi(m,s)\right| \le C(\Lambda^{m-s}m^{-\beta} + \Lambda_1^m),$$

$a_r(m,s)$ being independent of ρ. Break sum (3.26) into two: $\sum_{2T}^{M} = \sum_{2T}^{M_1} + \sum_{M_1}^{M}$, where $M_1, 2T \le M_1 \le M$ to be choosen later. For the first sum we have

$$\sum_{s=1}^{T} \sum_{m=2T}^{M_1} a_r(m,s)\xi(m,s)\cos(2\phi(m,s)) = O(TM_1^{1-2\beta}).$$

To estimate the second one we apply (3.9) with $L = \Lambda$, $k = 1 \pm \Lambda^{-s}$:

$$\begin{aligned}
&\mathrm{mes}\{\rho \in I : \sum_{s=1}^{T} \sum_{m=M_1}^{M} a_r(m,s)\xi(m,s)\cos(2\phi(m,s)) \ge \delta\} \\
&\le e^{-\delta t} \int_I d\rho \exp\left(t \sum_{s=1}^{T} \sum_{m=M_1}^{M} a_r(m,s)\xi(m,s)\cos(2\phi(m,s)\right) \\
&\le e^{-\delta t} \prod_{s=1}^{T} \left(\int_I d\rho \exp\left(Tt \sum_{m=M_1}^{M} a_r(m,s)\xi(m,s)\cos(2\phi(m,s)\right)\right)^{\frac{1}{T}} \\
&\le e^{-\delta t} \exp\left(CT^2t^2 \sum_{M_1}^{M} m^{-4\beta} + Ct \sum_{M_1}^{M} m^{-3\beta}\right),
\end{aligned}$$

provided

$$CTM_1^{-2\beta}t \le 1.$$

As a consequence, for δ satisfying

$$C\sum_{M_1}^{M} m^{-3\beta} \le \delta \le CTM_1^{2\beta} \sum_{M_1}^{M} m^{-4\beta},$$

one has

$$\begin{aligned}
&\mathrm{mes}\{\rho \in I : \sum_{s=1}^{T} \sum_{m=M_1}^{M} a_r(m,s)\xi(m,s)\cos(2\phi(m,s)) \ge \delta\} \\
&\le e^{-C\frac{\delta^2}{T^2\sum_{M_1}^{M} m^{-4\beta}}}.
\end{aligned}$$

Therefore, one can get the desired estimate (3.25) by choosing M_1 and δ exactly in the same way as M_1 and δ in the proof of lemma 3.2. ■

Combining (3.22) and lemmas 3.5, 3.6, one gets

Lemma 3.7. *For any α and a.e. ρ,*

$$\lim_{m\to+\infty}\frac{\Sigma_1(m)}{m^{1-2\beta-\varepsilon}}=0,$$

for some $\varepsilon>0$.

This lemma together with lemma 3.3 lead to proposition 3.1.

3.2. Decaying solutions of (3.1).

In this subsection we construct a decaying solution to (3.1). Consider the solutions

$$\chi_0(m)=R_0(m)\begin{pmatrix} e^{i\varphi_0(m)}\\ e^{-i\varphi_0(m)}\end{pmatrix},\quad \chi_{\pi/2}(m)=R_{\pi/2}(m)\begin{pmatrix} e^{i\varphi_{\pi/2}(m)}\\ e^{-i\varphi_{\pi/2}(m)}\end{pmatrix}.$$

One has

$$\det(\chi_0(m),\chi_{\pi/2}(m))=-2i,$$

which means that

$$|\zeta_0(m)-\zeta_{\pi/2}(m)|=\frac{2}{R_0(m)R_{\pi/2}(m)},$$

where $\zeta_\alpha(m)=e^{2i\varphi_\alpha(m)}$, $\alpha=0,\pi/2$. Applying the results of the previos subsection one gets for a.e. ρ

$$|\zeta_0(m)-\zeta_{\pi/2}(m)|=e^{-2r_*m^{1-2\beta}(1+o(1))},\quad m\to+\infty. \tag{3.27}$$

Set

$$v(m)=\ln\left(\frac{R_0(m)}{R_{\pi/2}(m)}\right).$$

By (3.2), (3.7) $v(m)$ satisfies

$$|v(m+1)-v(m)|\le Cm^{-\beta}|\zeta_0(m)-\zeta_{\pi/2}(m)|\le e^{-2r_*m^{1-2\beta}(1+o(1))}. \tag{3.28}$$

So, $v(m)$ has a limit v_∞ as $m\to+\infty$, and

$$|v(m)-v_\infty|\le e^{-2r_*m^{1-2\beta}(1+o(1))}. \tag{3.29}$$

It folows from (3.1) that $z(m)=e^{i(\varphi_0(m)-\varphi_{\pi/2}(m))}$ satisfies

$$\frac{z(m+1)}{z(m)}=\frac{R_0(m)R_{\pi/2}(m+1)(s(d_m)+\bar r(d_m)e^{-2i\Gamma_-(m)}\zeta_0(m))}{R_0(m+1)R_{\pi/2}(m)(s(d_m)+\bar r(d_m)e^{-2i\Gamma_-(m)}\zeta_{\pi/2}(m))}.$$

Combining this representation with (3.27), (3.28) one gets

$$|z(m+1)-z(m)|\le e^{-2r_*m^{1-2\beta}(1+o(1))}.$$

This means that $z(m)$ has a limit z_∞ and

$$|e^{i\varphi_0(m)}-z_\infty e^{i\varphi_{\pi/2}(m)}|\le e^{-2r_*m^{1-2\beta}(1+o(1))}. \tag{3.30}$$

Notice that by (3.27) $z_\infty^2=1$.

Proposition 3.1 and (3.29), (3.30) lead immediately to the following result.

Proposition 3.2. *For a.e ρ there exists a real constant h $(=-z_\infty e^{v_\infty})$ such that the solution $\chi_0(m)+h\chi_{\pi/2}(m)$ satisfies*

$$|\chi_0(m)+h\chi_{\pi/2}(m)|\le e^{-r_*m^{1-2\beta}(1+o(1))},$$

4. End of the proof of theorem 1.2

4.1. Growing and decaying solutions of (2.60).

Let us consider the solution Q^α of (2.1) corresponding to the following initial data:

$$Q^\alpha(\xi(x))|_{x=x_{qk_M}} = R_\alpha(M)\sin\Big(\xi(x_{qk_M}) + \varphi_\alpha(M)\Big),$$

$$Q^\alpha_\xi(\xi(x))|_{x=x_{qk_M}} = R_\alpha(M)\cos\Big(\xi(x_{qk_M}) + \varphi_\alpha(M)\Big),$$

where M is sufficiently large number, $\chi_\alpha(m) = R_\alpha(m)\binom{e^{i\varphi_\alpha(m)}}{e^{-i\varphi_\alpha(m)}}$ is a solution of (3.1), $\chi_\alpha(M_0) = \binom{e^{i\alpha}}{e^{-i\alpha}}$, M_0 being the same as in section 3. We denote by $\hat\chi_\alpha(m)$ the $\hat\chi(m)$ corresponding to Q^α:

$$\hat\chi_\alpha(m) = \tilde R(k_m)\begin{pmatrix} e^{i\tilde\varphi(k_m)} \\ e^{-i\tilde\varphi(k_m)}\end{pmatrix} = R(x_{qk_m})\begin{pmatrix} e^{i\varphi(x_{qk_m})} \\ e^{-i\varphi(x_{qk_m})}\end{pmatrix},$$

R, φ being the Prüfer coordinates associated to Q^α

$$Q^\alpha = R\sin(\xi+\varphi), \quad Q^\alpha_\xi = R\cos(\xi+\varphi).$$

One has the following proposition.

Proposition 4.1. *For any α and a.e ρ there exist real constants $g^1_\alpha \neq 0$, g^2_α such that as $m\to+\infty$,*

$$\hat\chi_\alpha(m) = g^1_\alpha\chi_\alpha(m) + g^2_\alpha\chi^d(m) + O(e^{-\gamma m}),$$

provided M is chosen sufficiently large (it may depend on ρ). Here $\chi^d(m)$ is the decaying solution of (3.1) *introduced in proposition* 3.1: $\chi^d(m) = \chi_0(m) + h\chi_{\pi/2}(m)$.

Proof. $\hat\chi_\alpha(m)$ satisfies

$$\hat\chi_\alpha(m+1) = \mathcal{A}_0(m)\hat\chi_\alpha(m) + \mathcal{R}(m), \quad \hat\chi_\alpha(M) = \chi_\alpha(M), \tag{4.1}$$

where

$$|\mathcal{R}(m)| \le Ce^{-\gamma m}|\hat\chi_\alpha(m)|.$$

We apply to (4.1) the following variation parameter type transformation:

$$\hat\chi_\alpha(m) = \Psi(m)g(m), \quad \Psi(m) = (\chi_\alpha(m), \chi^d(m)),$$

$\det\Psi(m)\neq 0$ for a.e. ρ. This transformation brings (4.1) to the form

$$g(m+1) = g(m) + \tilde{\mathcal{R}}(m), \quad \tilde{\mathcal{R}}(m) = \Psi^{-1}(m)\mathcal{R}(m), \quad g(M) = \begin{pmatrix}1\\0\end{pmatrix}. \tag{4.2}$$

Clearly,

$$|\tilde{\mathcal{R}}(m)| \le Ce^{-\gamma m}|g(m)|.$$

One can rewrites (4.2) in the form

$$g(m+1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=M}^{m} \tilde{\mathcal{R}}(j), \quad m \geq M.$$

As a consequence, for M sufficiently large, one has

$$|g(m) - \begin{pmatrix} 1 \\ 0 \end{pmatrix}| \leq Ce^{-\gamma M}, \quad m \geq M,$$

which, in particular, implies that

$$|g(m+1) - g(m)| \leq Ce^{-\gamma m}.$$

So, as $m \to +\infty$, $g(m)$ has a limit $g_\infty = \binom{g_\alpha^1}{g_\alpha^2}$ and

$$|g(m) - g_\infty| \leq Ce^{-\gamma m},$$
$$|g_\alpha^1 - 1|, |g_\alpha^2| \leq Ce^{-\gamma M}.$$

Returning to $\hat{\chi}_\alpha$ one gets

$$\hat{\chi}_\alpha(m) = \Psi(m) g_\infty + O(e^{-\gamma m}).$$

Note that $\overline{g(m)} = g(m)$, so, g_α^j are real. ■

The decaying solution of (2.60) can be now constructed as follows. Consider the solution Q^d of (2.2) defined by

$$Q^d = Q^0 + h_1 Q^{\pi/2},$$

where $h_1 = \frac{h g_0^1}{g_{\pi/2}^1} \in \mathbb{R}$. Then the corresponding $\hat{\chi}(m)$ has the form

$$\hat{\chi}(m) = \hat{\chi}_0(m) + h_1 \hat{\chi}_{\pi/2}(m),$$

and by propositions 4.1, admits the estimate

$$\hat{\chi}(m) = h_2 \chi^d(m) + O(e^{-\gamma m}),$$

for some constant h_2. In particular,

$$|\hat{\chi}(m)| \leq e^{-r_* m^{1-2\beta}(1+o(1))}. \tag{4.3}$$

Notice also that

$$\hat{\chi}(M) = \chi_0(M) + h\chi_{\pi/2}(M) \neq 0,$$

which means that Q^d is nontrivial solution.

We are now able to complete the proof of theorem 1.2. Let R_d and R_i be R's associated to Q^d and Q^0 respectively. Combining propositions 3.1, 3.2, 4.1 and (4.3), (2.52), (2.51), (2.30), (2.13), (2.7) one gets the following result.

Proposition 4.2. *For a.e.ρ, R_d, R_i satisfy*

$$\int_0^N dx p^{-1} R_d^2 \leq N^{1/2} e^{-\mu_* (\ln N)^{1-2\beta}(1+o(1))},$$

$$\int_0^N dx p^{-1} R_i^2 \geq N^{1/2} e^{\mu_* (\ln N)^{1-2\beta}(1+o(1))},$$

provided N is sufficiently large. Here $\mu_* = \frac{r_*}{(2\ln\Lambda)^{1-2\beta}}$.

This means in particular that for a.e ρ, Q^d is a subordinate solution of (2.2), which completes the proof of theorem 1.2.

Appendix 1

In this appendix we prove lemmas 2.1, 2.2, 2.3.

Proof of lemma 2.1. First we consider the integral

$$J=\int_{x_l}^{x} dy\psi_l(y)e^{2i\xi(y)}v_l(y),$$

where $\psi_l(x)$ stands for either $\chi_l(x)$ or $1-\chi_l(x)$, $x_l\le x\le x_{l+1}$, $v_l(x)=v(x)-\hat v(l)e^{-2i\pi lx}$, $v\in L_1(\mathbb{T})$, χ_l being described in lemma 2.1. It admits the estimate

$$|J|\le c\|v\|_{L_1(\mathbb{T})}. \tag{A1.1}$$

Indeed, for $|x-x_l|\le 4$, (A1.1) is trivial. For $|x-x_l|\ge 4$ we write the integral J as the sum

$$J=J^{(0)}+J^{(1)},$$

$$J^{(0)}=\int_{x_l}^{x} dye^{2i\xi(y)}(\chi(y-x_l)+\chi(y-x)\psi_l(y)v_l(y),$$

$$J^{(1)}=\int_{x_l}^{x} dye^{2i\xi(y)}(1-\chi(y-x_l)-\chi(y-x))\psi_l(y)v_l(y).$$

Clearly,

$$|J^{(0)}|\le c\|v_l\|_{L_1(\mathbb{T})}\le c\|v\|_{L_1(\mathbb{T})}. \tag{A1.2}$$

To estimate $J^{(1)}$ we represent it as the sum

$$J^{(1)}=\sum_{n,n\ne l}\hat v_n\zeta_n,$$

$$\zeta_n=\int_{x_l}^{x} dy(1-\chi(y-x_l)-\chi(y-x))\psi_l(y)e^{2i(\xi(y)-\pi ny)}.$$

Integrating by parts one gets the representation

$$\zeta_n=-\frac14\int_{x_l}^{x} dye^{2i(\xi(y)-\pi ny)}\left(\frac{d}{dy}\frac{1}{(p(y)-\pi n)}\right)^2(1-\chi(y-x_l)-\chi(y-x))\psi_l(y),$$

$$p(y)=(Fy+q(y)+E)^{1/2},$$

which leads to the estimate

$$|\zeta_n|\le c<n-l>^{-2}, n\ne l.$$

As a consequence,

$$|J^{(1)}|\le c\|v\|_{L_1(\mathbb{T})}. \tag{A1.3}$$

Combining (A1.2, A1.3), we get (A1.1).

Consider the integrals

$$J_1=\int^{x} dye^{2i(\xi(y)-\pi ly)}(1-\chi_l(y)),$$

$$J_2 = \int_{x_l}^{x} dy e^{2i(\xi(y)-\pi l y)} \chi_l(y).$$

Since for $|x - X_l| \geq cl^{1-\nu}$,

$$|\xi(x) - \pi l| \geq l^{-\nu},$$

the integration by parts in the first one gives immediatly

$$|J_1| \leq cl^{\nu}. \tag{A1.4}$$

The second integral can be represented in the form

$$J_2 = e^{2i(\xi(X_l)-\pi l X_l)} \int_{x_l}^{x} dy e^{i\mu_1(l)(y-X_l)^2 + i\mu_2(l)(y-X_l)^3} \chi_l(y) + O(l^{-3\nu}), \tag{A1.5}$$

where

$$\mu_1(l) = \frac{F}{2\pi l}, \quad \mu_2(l) = -\frac{F^2}{12(\pi l)^3}.$$

(A1.5) implies directly

$$|J_2| \leq cl^{1/2}. \tag{A1.6}$$

Combining (A1.1), (A1.4), (A1.6) one obtains lemma 2.1. ■

Proof of lemma 2.2. First we remark that up to the terms of order $O(l^{1/2})$ the expression $I_l(E)$ can be replaced by $I_l^*(E)$,

$$I_l^*(E) = \int\limits_{x_l^*}^{x_{l+1}^*} dy \int\limits_{y}^{x_{l+1}^*} ds e^{2i(\xi(s)-\xi(y))} v(s)\overline{v(y)},$$

where x_l^* defined by

$$Fx_l^* + q(x_l^*) = \pi^2(l - 1/2)^2.$$

Indeed, one has $x_l^* - x_l = O(1)$, which together with lemma 2.1 implies

$$I_l(E) = I_l^*(E) + O(l^{1/2}). \tag{A1.7}$$

To estimate $I_l^*(E)$ we write it as the sum

$$I_l^*(E) = \mathcal{I}_l^0(E) + \mathcal{I}_l^1(E) + \mathcal{I}_l^2(E) + \mathcal{I}_l^3(E), \tag{A1.8}$$

$$\mathcal{I}_l^0(E) = |\hat{v}(l)|^2 \int_{x_l^*}^{x_{l+1}^*} dy \int_{y}^{x_{l+1}^*} ds e^{2i(\xi(s)-\xi(y)-\pi l(s-y))}$$

$$\mathcal{I}_l^1(E) = \int_{x_l^*}^{x_{l+1}^*} dy \int_{y}^{x_{l+1}^*} ds e^{2i(\xi(s)-\xi(y))} v_l(s)\overline{v_l(y)}$$

$$\mathcal{I}_l^2(E) = \overline{\hat{v}(l)} \int_{x_l^*}^{x_{l+1}^*} dy \int_{y}^{x_{l+1}^*} ds e^{2i(\xi(s)-\xi(y)+\pi l y)} v_l(s)$$

$$\mathcal{I}_l^3(E) = \hat{v}(l) \int_{x_l^*}^{x_{l+1}^*} dy \int_{y}^{x_{l+1}^*} ds e^{2i(\xi(s)-\xi(y)-\pi l s)} \overline{v_l(y)}$$

$$= \hat{v}(l) \int_{x_l^*}^{x_{l+1}^*} dy \int_{x_l^*}^{y} dse^{2i(\xi(y)-\xi(s)-\pi l y)} \overline{v_l(s)}. \tag{A1.9}$$

It follows from (A1.1) that for $x_l^* \le y \le x_{l+1}^*$,

$$|\int_{y}^{x_{l+1}^*} dse^{2i\xi(s)} v_l(s)| \le c\|v\|_1.$$

As a consequence,

$$|\mathcal{I}_l^j(E)| \le Cl, \quad j = 1,2,3. \tag{A1.10}$$

Consider the derivatives $\frac{d\mathcal{I}_l^j}{dE}$, $j = 1,2$. We write them as

$$\frac{d\mathcal{I}_l^1}{dE}(E) = i \int\limits_{x_l^*}^{x_{l+1}^*} dy \int\limits_{y}^{x_{l+1}^*} dse^{2i(\xi(s)-\xi(y))} \left(\int\limits_{x_l^*}^{s} d\rho p^{-1}(\rho, E) - \int\limits_{x_l^*}^{y} d\rho p^{-1}(\rho, E) \right) v_l(s)\overline{v_l(y)},$$

$$\frac{d\mathcal{I}_l^2}{dE}(E) = i\overline{\hat{v}(l)} \int\limits_{x_l^*}^{x_{l+1}^*} dy \int\limits_{y}^{x_{l+1}^*} dse^{2i(\xi(s)-\xi(y)+\pi l y)} \left(\int\limits_{x_l^*}^{s} d\rho p^{-1}(\rho, E) \right.$$

$$\left. - \int\limits_{x_l^*}^{y} d\rho p^{-1}(\rho, E) \right) v_l(s).$$

Clearly, for $x_l^* \le y \le x_{l+1}^*$ one has

$$\left| \int_{x_l^*}^{y} d\rho p^{-1}(\rho, E) \right| \le c,$$

$$|\int_{y}^{x_{l+1}^*} dse^{2i\xi(s)} \int_{x_l^*}^{s} d\rho p^{-1}(\rho, E) v_l(s)| \le c\|v\|_1.$$

As a consequence,

$$\left| \frac{d\mathcal{I}_l^j}{dE} \right| \le Cl, \quad j = 1,2. \tag{A1.11}$$

By (A1.9), the same estimate is valid for $\frac{d\mathcal{I}_l^3}{dE}$

$$\left| \frac{d\mathcal{I}_l^3}{dE} \right| \le Cl. \tag{A1.12}$$

To analyse the expression $\mathcal{I}_l^0$ we represent it as

$$\mathcal{I}_l^0 = |\hat{v}(l)|^2 (\mathcal{I}_l^{00} + \mathcal{I}_l^{01} + \mathcal{I}_l^{02}),$$

$$\mathcal{I}_l^{00} = \int^{x_{l+1}^*} dy \tilde{\chi}_l(y) \int^{x_{l+1}^*} ds \chi_l(s) e^{2i(\xi(s)-\xi(y)-\pi l(s-y))},$$

$$\mathcal{I}_l^{01} = \int_{x_l^*}^{x_{l+1}^*} dy\tilde{\chi}_l(y) \int_y^{x_{l+1}^*} ds(1-\chi_l(s))e^{2i(\xi(s)-\xi(y)-\pi l(s-y))},$$

$$\mathcal{I}_l^{02} = \int_{x_l^*}^{x_{l+1}^*} dy(1-\tilde{\chi}_l(y)) \int_y^{x_{l+1}^*} dse^{2i(\xi(s)-\xi(y)-\pi l(s-y))}.$$

Here $\chi_l(x) = \chi(l^{-1+\nu}(x-X_l))$, $\tilde{\chi}_l(x) = \chi(l^{-1+\tilde{\nu}}(x-X_l))$, $0<\nu<\tilde{\nu}<1/2$.

First we consider $\mathcal{I}_l^{02}$. Integrating by parts and using (A1.4), (A1.6) one gets

$$\mathcal{I}_l^{02} = \frac{i}{\pi}\int_{x_l^*}^{x_{l+1}^*} dse^{2i(\xi(s)-\pi l s)} + O(l^{-1/2})$$

$$-\frac{i}{2}\int_{x_l^*}^{x_{l+1}^*} dye^{-2i(\xi(y)-\pi l y)}\frac{d}{dy}\left(\frac{1}{p(y)-\pi l}(1-\tilde{\chi}_l(y))\int_y^{x_{l+1}^*} dse^{2i(\xi(s)-\pi l s)}\right)$$

$$= O(l^{1/2+\tilde{\nu}}) + \frac{i}{2}\int_{x_l^*}^{x_{l+1}^*} dy\frac{1-\tilde{\chi}_l(y)}{p(y)-\pi l}.$$

Consider the integral

$$\int_{x_l^*}^{x_{l+1}^*} dy\frac{1-\tilde{\chi}_l(y)}{p(y)-\pi l}. \tag{A1.13}$$

One has

$$(A1.13) = \left(\int_{x_l^*}^{X_l-2l^{1-\tilde{\nu}}} dy + \int_{X_l+2l^{1-\tilde{\nu}}}^{x_{l+1}^*} dy\right)\frac{1}{p(y)-\pi l}$$

$$+\frac{2\pi l}{F+q'(X_l)}\int\limits_{|y-X_l|\le 2l^{1-\tilde{\nu}}} dy\frac{1-\tilde{\chi}_l(y)}{y-X_l} + O(l^{-\tilde{\nu}}).$$

Since χ is an even function the last integral here vanishs. Therefore, one has

$$(A1.13) = \left(\int_{x_l^*}^{X_l-2l^{1-\tilde{\nu}}} dy + \int_{X_l+2l^{1-\tilde{\nu}}}^{x_{l+1}^*} dy\right)\frac{1}{p(y)-\pi l} + O(l^{-\tilde{\nu}})$$

$$= \frac{2}{F}\left(\int_{x_l^*}^{X_l-2l^{1-\tilde{\nu}}} dy + \int_{X_l+2l^{1-\tilde{\nu}}}^{x_{l+1}^*} dy\right)\frac{p'p}{p-\pi l} + O(1) = O(1).$$

As a consequence,

$$|\mathcal{I}_l^{02}| \le Cl^{1-\gamma}. \tag{A1.14}$$

Consider $\mathcal{I}_l^{01}$. By (A1.4),

$$|\mathcal{I}_l^{01}| \le Cl^{1+\nu-\tilde{\nu}}. \tag{A1.15}$$

The expression $\mathcal{I}_l^{00}$ admits the representation

$$\mathcal{I}_l^{00} = \int^{x_{l+1}^*} dy\tilde{\chi}_l(y)\int^{x_{l+1}^*} ds\chi_l(s)e^{i\mu_1(l)(X_l)((s-X_l)^2-(y-X_l)^2)} + O(l^{2-4\nu-\tilde{\nu}})$$

$$= \int_{-\infty}^{\infty} dy\chi(l^{-1+\tilde{\nu}}y) \int_y^\infty ds\chi(l^{-1+\nu}s)e^{i\mu_1(l)(s^2-y^2)} + O(l^{2-4\nu-\tilde{\nu}})$$

$$= 2\int_0^\infty dy\chi(l^{-1+\tilde{\nu}}y)e^{-i\mu_1(l)y^2} \int_0^\infty ds\chi(l^{-1+\nu}s)e^{i\mu_1(l)s^2} + O(l^{2-4\nu-\tilde{\nu}}). \quad \text{(A1.16)}$$

At the last step here we used the fact that χ is an even function. The expression (A1.16) allows some further simplifications

$$(A1.16) = \frac{\pi^2 l}{F} + O(l^{1/2+\tilde{\nu}}) + +O(l^{2-4\nu-\tilde{\nu}}).$$

Choosing

$$\frac{1}{5} < \nu < \tilde{\nu} < \frac{1}{2},$$

one gets

$$\mathcal{I}_l^{00} = \frac{\pi^2 l}{F} + O(l^{1-\gamma}), \quad \text{(A1.17)}$$

or combining (A1.14), (A1.15), (A1.16)

$$\mathcal{I}_l^0 = |\hat{v}(l)|^2 \frac{\pi^2 l}{F} + O(l^{1-\gamma}), \quad \text{(A1.18)}$$

which together with (A1.7), (A1.8), (A1.10), (A1.11), (A1.18) gives the representation of lemma 2.2, $\mathcal{I}_l(E)$ being given by

$$\mathcal{I}_l(E) = |\hat{v}(l)|^2 \frac{\pi^2 l}{F} + \mathcal{I}_l^1(E) + \mathcal{I}_l^2(E) + \mathcal{I}_l^3(E). \quad \text{(A1.19)}$$

■

Proof of lemma 2.3. Consider the integral

$$\int_{x_l}^{x_{l+1}} dy e^{2i\xi(y)} v(y). \quad \text{(A1.20)}$$

We break it into three parts

$$\int_{x_l}^{x_{l+1}} dy = \int_{x_l}^{L_l^- +1/2} dy + \int_{L_l^+ -1/2}^{x_{l+1}} dy + \int_{L_l^- +1/2}^{L_l^+ -1/2} dy,$$

where $L_l^\pm = [X_l \pm l^{1-\nu}]$, $0 < \nu < 1/2$ to be fixed later. We start by estimating the two first integrals here. Write them as the sums

$$\int_{x_l}^{L_l^- +1/2} dy e^{2i\xi(y)} v(y) + \int_{L_l^+ -1/2}^{x_{l+1}} dy e^{2i\xi(y)} v(y)$$

$$= \left(\sum_{n=x_l+1/2}^{L_l^-} + \sum_{L_l^+}^{x_{l+1}-1/2} \right) \int_{n-1/2}^{n+1/2} dy e^{2i\xi(y)} v(y). \quad \text{(A1.21)}$$

For $|n-X_l|\geq Cl^{1-\nu}$, the expression $\int\limits_{n-1/2}^{n+1/2} dye^{2i\xi(y)}v(y)$ has the form

$$\int\limits_{n-1/2}^{n+1/2} dye^{2i\xi(y)}v(y)=e^{2i\xi(n)}\int_{-1/2}^{1/2} dye^{2i\xi'(n)y}(1+i\xi''(n)y^2)v(y)+O(l^{-2}). \quad \text{(A1.22)}$$

Under assumptions (2.12) one has for any integer m, $m\geq 1$, $k\in\mathbb{R}$,

$$\int_{-1/2}^{1/2} dye^{2iky}y^m v(y)=O(k^{-1}). \quad \text{(A1.23)}$$

So, the r.h.s. of (A1.22) can be simplified:

$$\int\limits_{n-1/2}^{n+1/2} dye^{2i\xi(y)}v(y)=e^{2i\xi(n)}V(n)+O(l^{-2}),\quad V(n)=\hat{v}\Big(\frac{\xi'(n)}{\pi}\Big),$$

$\hat{v}(k)=\int_{-1/2}^{1/2} dye^{2ik\pi y}v(y)$. As a consequence, one obtains

$$(A1.21)=\Big(\sum_{n=x_l+1/2}^{L_l^-}+\sum_{n=L_l^+}^{x_{l+1}-1/2}\Big)e^{2i\xi(n)}V(n)+O(l^{-1})$$

$$=e^{2i\xi(L_l^-)}\frac{V(L_l^-)}{1-e^{-2i\xi^{(1)}(L_l^-)}}-e^{2i\xi(L_l^+-1)}\frac{V(L_l^+)}{1-e^{-2i\xi^{(1)}(L_l^+)}} \quad \text{(A1.24)}$$

$$+e^{2i\xi(x_{l+1}-1/2)}\frac{V(x_{l+1}-1/2)}{1-e^{-2i\xi^{(1)}(x_{l+1}-1/2)}}-e^{2i\xi(x_l-1/2)}\frac{V(x_l+1/2)}{1-e^{-2i\xi^{(1)}(x_l+1/2)}} \quad \text{(A1.25)}$$

$$+\Big(\sum_{n=x_l+1/2}^{L_l^--1}+\sum_{n=L_l^+}^{x_{l+1}-\frac{3}{2}}\Big)e^{2i\xi(n)}V_1(n)+O(l^{-1}). \quad \text{(A1.26)}$$

Here

$$\xi^{(1)}(n)=\xi(n)-\xi(n-1),\quad V_1(n)=\frac{V(n)}{1-e^{-2i\xi^{(1)}(n)}}-\frac{V(n+1)}{1-e^{-2i\xi^{(1)}(n+1)}}.$$

Clearly, $V_1(n)$ satisfies

$$|V_1(n)|\leq Cl^{-1+2\nu},$$

$$\left|\frac{V_1(n)}{1-e^{-2i\xi^{(1)}(n)}}-\frac{V_1(n+1)}{1-e^{-2i\xi^{(1)}(n+1)}}\right|\leq Cl^{-2+2\nu}(\sin(p(n))^{-2}.$$

So, repeating the procedure one gets

$$\Big(\sum_{n=x_l+1/2}^{L_l^--1}+\sum_{n=L_l^+}^{x_{l+1}-3/2}\Big)e^{2i\xi(n)}V_1(n)=O(l^{-1+3\nu}). \quad \text{(A1.27)}$$

Consider the expressions (A1.24), (A1.25). The first one may be represented as

$$(A1.24) = i\frac{\pi}{F}\hat{v}_l l^{\nu}\big(e^{i\xi(L_l^+-1)+\frac{F}{2\pi}l^{-\nu}} + e^{i\xi(L_l^-)-\frac{F}{2\pi}l^{-\nu}}\big) + O(l^{-\gamma}). \tag{A1.28}$$

Expression (A1.25) has the form

$$(A1.25) = t_{l+1} - t_l + O(l^{-1}), \tag{A1.29}$$

where

$$t_l = e^{2i\xi(x_l-1/2)}\frac{V(x_l-1/2)}{1-e^{-2i\xi^{(1)}(x_l-1/2)}}.$$

Clearly,

$$|t_l| \le C.$$

Combining (A1.27), (A1.28), (A1.29), one obtains

$$\begin{aligned}(A1.21) = {} & i\frac{\pi}{F}\hat{v}_l l^{\nu}\big(e^{i\xi(L_l^+-1)+\frac{F}{2\pi}l^{-\nu}} + e^{i\xi(L_l^-)-\frac{F}{2\pi}l^{-\nu}}\big) \\ & + t_{l+1} - t_l + O(l^{-1+3\nu}) + O(l^{-\gamma}).\end{aligned} \tag{A1.30}$$

Next we consider the expression

$$\int_{L_l^-+1/2}^{L_l^+-1/2} dy e^{2i\xi(y)}v(y) = \sum_{n=L_l^-+1}^{L_l^+-1}\int_{n-1/2}^{n+1/2} dy e^{2i\xi(y)}v(y).$$

For $|n-X_l| \le Cl^{1-\nu}$, one has

$$\int_{n-1/2}^{n+1/2} dy e^{2i\xi(y)}v(y) = e^{2i\xi(X_l)-2i\pi l X_l}e^{i\mu_1(l)(n-X_l)^2}\hat{v}_l + O(l^{-3\nu}).$$

We fix now ν in such a way that

$$1/4 < \nu < 1/3.$$

As a consequence, one obtains

$$\begin{aligned}\int_{L_l^-+1/2}^{L_l^+-1/2} dy e^{2i\xi(y)}v(y) &= e^{2i\xi(X_l)-2i\pi l X_l}\sum_{n=L_l^-+1}^{L_l^+-1} e^{i\mu_1(l)(X_l)(n-X_l)^2}\hat{v}_l + O(l^{-\gamma}) \\ &= e^{2i\xi(X_l)-2i\pi l X_l}\hat{v}_l\int_{L_l^-+1}^{L_l^+} dy e^{i\mu_1(l)(y-X_l)^2} + O(l^{-\gamma}).\end{aligned} \tag{A1.31}$$

The integral in the r.h.s may be represented as

$$\begin{aligned}e^{2i\xi(X_l)-2i\pi l X_l}\hat{v}_l\int_{L_l^-+1}^{L_l^+} dy e^{i\xi''(X_l)(y-X_l)^2} &= e^{2i\omega_l}\pi\left(\frac{2l}{F}\right)^{1/2} \\ -i\frac{\pi}{F}\hat{v}_l l^{\nu}&\left(e^{2i\xi(L_l^+)} + e^{2i\xi(L_l^-+1)}\right) + O(l^{-\gamma}).\end{aligned} \tag{A1.32}$$

Combining (A1.30), (A1.31), (A1.32) one gets lemma 2.3. ■

Appendix 2

Here we prove lemma 3.1. The proof is based on the nequality:

$$e^z \le 1 + z + Bz^2, \tag{A2.1}$$

valid, say, for $|z| \le 1$. By (A2.1),

$$\int_I dy \exp(tS_N(y)) \le X_N(t), \quad X_N(t) = \int_I dy \mathcal{X}_N(y,t), \tag{A2.2}$$

$$\mathcal{X}_N(y,t) = \prod_{n=0}^{N} (1 + ts_n(y) + Bt^2 s_n^2(y)),$$

$$s_n(y) = a(n) f_n(L^{h+n}y) g(L^{h+n}y),$$

provided

$$t \max_{0 \le n \le N} |a(n)| \le 1. \tag{A2.3}$$

We represent $X_N(t)$ as follows

$$X_N(t) = X_{N-1}(t) + (\mathrm{I}) + (\mathrm{II}),$$

$$(\mathrm{I}) = Bt^2 a^2(N) \int_I dy \mathcal{X}_{N-1}(y,t) f_N^2(L^{h+N}y) g^2(L^{h+R}y)$$

$$(\mathrm{II}) = ta(N)C(t), \quad C(t) = \int_I dy \mathcal{X}_{N-1}(y,t) f_N(L^{h+N}y) g(L^{h+N}y).$$

For (I) one has the obvious estimate

$$|(\mathrm{I})| \le Bt^2 a^2(N) X_{N-1}(t). \tag{A2.4}$$

Consider expression (II). One can write $C(t)$ in the form

$$C(t) = C^1(t) + C^2(t),$$

$$C^1(t) = \int_I dy \mathcal{X}_{N-1}(y,t) f_N(L^{h+N}y)$$

$$\times \left(g(L^{h+N}y) - \int_I d\xi g(L^{h+N}\xi) \right)$$

$$C^2(t) = \left(\int_I d\xi g(L^{h+N}\xi) \right) \left(\int_I dy \mathcal{X}_{N-1}(y,t) f_N(L^{h+N}y) \right).$$

Let

$$g(L^{h+N}y) - \int_I d\xi g(L^{h+N}\xi) = h'(y),$$

where $h = 0$ at the end points of I. Clearly,

$$\left| \int_I d\xi g(L^{h+N}\xi) \right| \le 2KL^{-h-N},$$

$$|h(y)| \le 4KL^{-h-N}, \quad y \in I.$$

Therefore,

$$|C^2(t)| \le 2KL^{-h-N} X_{N-1}(t), \tag{A2.5}$$

$$|C^1(t)| \le B(K,\Lambda) X_{N-1}(t) \left(\|f'_N\|_\infty + L^{-h-N} t \sum_{j=0}^{N-1} \|s'_j\|_\infty \right)$$

$$\le B(K,\Lambda) X_{N-1}(t) \left(\|f'_N\|_\infty + t \sum_{j=0}^{N-1} |a(j)| L^{j-N} \right). \tag{A2.6}$$

Combining (A2.5), (A2.6) one gets

$$|(\mathrm{II})| \le B(K,\Lambda) X_{N-1}(t) t |a(N)|$$

$$\times \left(L^{-h-N} + \|f'_N\|_\infty + t \sum_{j=0}^{N-1} |a(j)| L^{j-N} \right), \tag{A2.7}$$

which together with (A2.4) leads to the inequality

$$X_N(t) \le X_{N-1}(t) \Big(1 + Bt^2 a^2(N)$$

$$+ B(K,\Lambda) t |a(N)| \big(L^{-h-N} + \|f'_N\|_\infty + t \sum_{j=0}^{N-1} |a(j)| L^{j-N} \big) \Big). \tag{A2.8}$$

In the same way one obtains

$$X_0(t) \le 1 + Bt^2 a^2(0) + B(K,\Lambda) t |a(0)| (L^{-h} + \|f'_0\|_\infty). \tag{A2.9}$$

Combining (A2.8), (A2.9) one gets

$$X_N(t) \le \exp\left(Bt^2 A^2(N) + B(K,\Lambda) \big(t\mathcal{Q}(N) + t^2 \sum_{n=1}^{N} \sum_{j=0}^{n-1} |a(n)||a(j)| L^{j-n} \big) \right)$$

$$\le \exp\left(B(K,\Lambda) t^2 A^2(N) + B(K,\Lambda) t \mathcal{Q}(N) \right),$$

where

$$A^2(N) = \sum_{n=1}^{N} a^2(n), \quad \mathcal{Q}(N) = \sum_{n=0}^{N} |a(n)| (L^{-h-n} + \|f'_n\|_\infty).$$

$\blacksquare$

Appendix 3

In this appendix we prove the following proposition.

Proposition A3.1. *Let $v \in L_1(\mathbb{T})$. Then the essential spectrum of operator* (1.1), *$\sigma_{ess}(H)$, fills up the real axis:*

$$\sigma_{ess}(H) = \mathbb{R}. \tag{A3.1}$$

Proof. We prove (A3.1) by employing the constructions of [16]. We start by replacing the whole line operator by a half-line operator by putting a Dirichlet condition at $x = R$. This is a relative trace class perturbation and it decomposes the whole-line operator into the direct sum of two half-line operators. Since the spectrum of the left half-line operator is discrete, one has

$$\sigma_{ess}(H) = \sigma_{ess}(H_R),$$

for any $R \in \mathbb{R}$. Here H_R denote right half-line operator with the Dirichlet condition as $x = R$

$$(H_R\psi)(x) = -\psi'' - (Fx + q(x) - v(x))\psi(x), \quad x \geq R, \quad \psi(R) = 0.$$

So, to prove (A3.1) it is sufficient to show that for some $R \in \mathbb{R}$ and $E \in \mathbb{C}$, $\operatorname{im} E > 0$, the difference

$$R_v(E) - R_0(E) \tag{A3.2}$$

is a compact operator. Here $R_v(E) = (H_R - E)^{-1}$, $R_0(E) = (H_0 - E)^{-1}$, H_0 stands for the free right half-line operator with the Dirichlet condition as $x = R$:

$$(H_0\psi)(x) = -\psi'' - (q(x) + Fx)\psi(x), \quad x \geq R, \quad \psi(R) = 0$$

To calculate (A3.2) we write the equation

$$(H_R - E)\psi = f, \quad \operatorname{im} E > 0$$

as a first-order system

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ v - q - Fx - E & 0 \end{pmatrix} \begin{pmatrix} \psi \\ \psi' \end{pmatrix} - \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{A3.3}$$

and apply a variation of parameter-type transformation that brings the system into nearly diagonalized form:

$$\begin{pmatrix} \psi \\ \psi' \end{pmatrix} = \mathbb{E}(x,E)\vec{z}, \quad \mathbb{E}(x,E) = \begin{pmatrix} 1 & 1 \\ \mathcal{E}(x,E) & \mathcal{E}^*(x,E) \end{pmatrix}, \; \vec{z} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

$$\mathcal{E} = \frac{f'_{as}}{f_{as}}, \quad \mathcal{E}^*(x,E) = \frac{f^{*\prime}_{as}}{f^*_{as}}, \tag{A3.4}$$

where $f_{as}(x,E)$, $f^*_{as}(x,E)$ are standard WKB asymptotics corresponding to the equation $-\psi_{xx} - (q(x) + Fx)\psi = E\psi$:

$$f_{as}(x,E) = p^{-1/2}(x,E)e^{i\Phi_{as}(x,E)}, \quad f^*_{as}(x,E) = p^{-1/2}(x,E)e^{-i\Phi_{as}(x,E)}$$

$$p(x,E) = (Fx - q + E)^{1/2}, \quad \Phi_{as}(x,E) = \int_R^x dsp(s,E),$$

R is supposed to be sufficiently large. The roots here are defined on the complex plane with the cut along negative imaginary semi-axes and is positive for positive values of the arguments.

Applying (A3.4) to (A3.3) we arrive at

$$\mathcal{H}_v(E)\vec{z} = \vec{g}, \quad \vec{g} = \frac{i}{2}\begin{pmatrix} f \\ -f \end{pmatrix}, \tag{A3.5}$$

$$\mathcal{H}_v(E) = \mathcal{H}_v^d(E) + \mathcal{V}.$$

Here $\mathcal{H}_v^d$ ($\mathcal{V}$) stands for the diagonal (anti-diagonal) part of the operator $\mathcal{H}_v$:

$$\mathcal{H}_v^d = p\left[\frac{d}{dx} - \begin{pmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E}^* \end{pmatrix}\right] + iV\sigma_3, \quad \mathcal{V} = V\sigma_2, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},$$

$$V = V_0 + \frac{1}{2}v, \quad V_0 = -\frac{5}{32}(F+q')^2 p^{-4} + \frac{1}{8}q'' p^{-2}.$$

We consider $\mathcal{H}_v(E)$ as an operator in $L_2([R,\infty) \to \mathbb{C}^2)$ submitted to the boundary condition: $z_1(R) + z_2(R) = 0$. We are going to treat the anti-diagonal part $\mathcal{V}$ as a perturbation. To this purpose, we rewrite (A3.5) in the form

$$z = \mathcal{A}_v g - \mathcal{B}_v g + \mathcal{B}_v \mathcal{V} z, \quad \mathcal{A}_v = \mathcal{H}_v^{d^{-1}}, \quad \mathcal{B}_v = \mathcal{A}_v \mathcal{V} \mathcal{A}_v,$$

which leads to the following representation for the resolvent R_v:

$$R_v = \mathbf{p}\mathbb{E}(\mathcal{A}_v J - \mathcal{B}_v J + \mathcal{B}_v \mathcal{V} \mathcal{R}_v), \tag{A3.6}$$

where $\mathbf{p}$ is the projection of $\mathbb{C}^2$-vector onto the first component: $\mathbf{p} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the operators J, $\mathcal{R}_v(E)$: $L_2([R,\infty) \to \mathbb{C}) \to L_2([R,\infty) \to \mathbb{C}^2)$ are given by the formulas

$$Jf = \frac{i}{2}\begin{pmatrix} f \\ -f \end{pmatrix}, \quad \mathcal{R}_v(E)f = \mathbb{E}^{-1}\begin{pmatrix} R_v(E)f \\ \frac{d}{dx}R_v(E) \end{pmatrix}.$$

Since $\mathbf{p}\mathbb{E}$, J are bounded in order to prove (A3.1) it is sufficient to show that $\mathcal{A}_v - \mathcal{A}_0$, $\mathcal{B}_v$, $\mathcal{B}_v \mathcal{V} \mathcal{R}_v$ are compact in L_2. Here $\mathcal{A}_0$ corresponds to $v = 0$:

$$\mathcal{A}_0 = \left[p\left(\frac{d}{dx} - \begin{pmatrix} \mathcal{E} & 0 \\ 0 & \mathcal{E}^* \end{pmatrix}\right) + iV_0\sigma_3\right]^{-1}.$$

Consider the operator $\mathcal{A}_v$. It has the form $\mathcal{A}_v = \begin{pmatrix} \mathcal{A}_v^1 & \mathcal{A}_v^2 \\ 0 & \mathcal{A}_v^3 \end{pmatrix}$, where $\mathcal{A}_v^j$, are integral operators with the kernels

$$\mathcal{A}_v^1(x,y) = a(x,y)\Theta(x-y), \quad \mathcal{A}_v^3(x,y) = -a(y,x)\Theta(y-x), \quad \mathcal{A}_v^2(x,y) = t(x)t(y),$$

$$a(x,y) = \frac{e^{i\int_y^x ds(p - Vp^{-1})}}{(p(x)p(y))^{1/2}}, \quad t(x) = \frac{e^{i\int_R^x ds(p - Vp^{-1})}}{(p(x))^{1/2}}, \tag{A3.7}$$

where Θ is the Heviside function. Clearly, for $\operatorname{im} E > 0$, one has

$$|a(x,y)| \leq Cx^{-1/4}y^{-1/4}e^{-\beta \operatorname{im} E(x^{1/2}-y^{1/2})}, \quad x \geq y \geq R, \tag{A3.8}$$

$$|t(x)| \leq Cx^{-1/4}e^{-\beta \operatorname{im} E(x^{1/2}-R^{1/2})}, \quad x \geq R, \tag{A3.9}$$

provided $\beta < F^{-1/2}$, R is sufficiently large. This allows us to conclude that $\mathcal{A}_v$ is bounded in $L_2([R,\infty))$ and, moreover, the component $\mathcal{A}_v^2$ is a Hilbert- Schmidt operator:

$$\|\mathcal{A}_v^2\|_{L_2([R,\infty)^2)} \leq C. \tag{A3.10}$$

Consider the difference $a(x,y) - a_0(x,y)$, where $a_0(x,y)$ corresponds to $v = 0$:

$$a_0(x,y) = \frac{e^{i\int_y^x ds(p - V_0 p^{-1})}}{(p(x)p(y))^{1/2}}. \tag{A3.11}$$

Due to (1.2) one has

$$|a(x,y) - a_0(x,y)| \leq |a_0(x,y)| \left| \int_y^x \frac{v(s)}{p(s,E)} ds \right|$$

$$\leq Cx^{-1/4}y^{-3/4}e^{-\beta \operatorname{im} E(x^{1/2}-y^{1/2})}, \quad x \geq y \geq R, \tag{A3.12}$$

which implies that

$$\|\mathcal{A}_v^j - \mathcal{A}_0^j\|_{L_2([R,\infty)^2)} \leq C, \quad j = 1,3. \tag{A3.13}$$

As a consequence of (A3.10), (A3.13) we obtain that the whole difference $\mathcal{A}_v - \mathcal{A}_0$ belongs to the Hilbert- Schmidt class.

Consider $\mathcal{B}_v$. Its kernel has the form: $\mathcal{B}_v(x,y) = \mathcal{C}_v(x,y) + \mathcal{D}_v(x,y)$,

$$\mathcal{C}_v(x,y) = \begin{pmatrix} \mathcal{C}_v^1(x,y) & \mathcal{C}_v^2(x,y) \\ 0 & \mathcal{C}_v^3(x,y) \end{pmatrix}, \quad \mathcal{D}_v(x,y) = \begin{pmatrix} 0 & \mathcal{D}_v^1(x,y) \\ \mathcal{D}_v^2(x,y) & 0 \end{pmatrix}$$

$$\mathcal{C}_v^1(x,y) = -it(x)t(y)\Psi^1(y), \quad \mathcal{C}_v^2(x,y) = -it(x)t(y)\Psi^1(R),$$

$$\mathcal{C}_v^3(x,y) = it(x)t(y)\Psi^1(x),$$

$$\mathcal{D}_v^1(x,y) = -ia(x,y)\Psi^2(y), \quad \mathcal{D}_v^2(x,y) = -ia(x,y)\Psi^1(x), \quad x \geq y \geq R,$$

$$\mathcal{D}_v^j(x,y) = \mathcal{D}_v^j(y,x), \quad j = 1,2.$$

Here

$$\Psi^1(x,E) = \int_x^\infty dy \frac{e^{2i\int_x^y ds(p - Vp^{-1})}}{p(y,E)} V(y),$$

$$\Psi^2(x,E) = \int_R^x dy \frac{e^{2i\int_y^x ds(p - Vp^{-1})}}{p(y,E)} V(y).$$

Clearly,

$$\|\Psi^i\| \leq C, \quad i = 1,2. \tag{A3.14}$$

which together with (A3.8), (A3.9) implies that $\mathcal{C}_v$, D_v are bounded operators from $l^2(L_1)$ to L_2:

$$\|\mathcal{C}_v g\|_{L_2([R,\infty))},\ \|\mathcal{D}_v g\|_{L_2([R,\infty))} \leq C\|g\|_{l^2(L_1)([R,\infty))}, \tag{A3.15}$$

provided R is sufficiently large. Here $l^p(L_q)$ norms are defined by

$$\|f\|_{l^p(L_q)([R,\infty))} = \left(\sum_{n=0}^{\infty} \|f\|^p_{L_q(\triangle_n)}\right)^{\frac{1}{p}}, \quad \triangle_n = [n+R, n+1+R].$$

Moreover, since $t(x)$, $t(x)\Psi^1(x)$ belong to $l^2(L_\infty)$, $\mathcal{C}_v$ is a compact operator. The same is true for $\mathcal{C}_v\mathcal{V}\mathcal{R}_v$. Indeed, $\mathcal{R}$ is a bounded operator from L_2 to $l^2(L_\infty)$:

$$\|\mathcal{R}_v g\|_{l^2(L_\infty)} \leq C\|g\|_{L_2}, \tag{A3.16}$$

see [16], for example. As a consequence, $\mathcal{C}_v\mathcal{V}\mathcal{R}_v$ is compact in L_2.

We next focus on the contribution of $\mathcal{D}_v$ in (A3.6). Consider the functions $\Psi^j(x,E)$, $j=1,2$. They admit the representations

$$\Psi^j(x,E) = \Psi^j_0(x,E) + O(x^{-1/2}), \tag{A3.17}$$

where

$$\Psi^1_0(x,E) = \int_x^\infty dy \frac{e^{2i\int_x^y ds(p_0+Ep_0^{-1})}}{p_0(y)} v(y),$$

$$\Psi^2_0(x,E) = \int_R^x dy \frac{e^{-2i\int_x^y ds(p_0+Ep_0^{-1})}}{p_0(y)} v(y),$$

$p_0(x) = p(x,0)$. It is not difficult to show that for $v \in L_1(\mathbb{T})$,

$$|\Psi^1_0(x,E)| \leq Cx^{-1/4}, \tag{A3.18}$$

see for example, [16], appendix 3. Due to the representation

$$\Psi^2_0(x,E) = -\overline{\Psi^1_0(x,E)} + e^{2i\int_R^x ds(p_0+Ep_0^{-1})}\overline{\Psi^1_0(E,R)}$$

$$+2\,\mathrm{im}\,E\int_R^x dy \frac{e^{2i\int_y^x ds(p_0+Ep_0^{-1})}}{p_0(y,E)}\Psi^1_0(y,E),$$

the same inequality is valid for $\Psi^2_0(x,E)$:

$$|\Psi^2_0(x,E)| \leq Cx^{-1/4}.$$

As a consequence, one gets the following estimate for $\mathcal{D}_v$:

$$\|\mathcal{D}_v g\|_{L_2([A,\infty))} \leq CA^{-1/4}\|g\|_{l^2(L_1)([R,\infty))}, \tag{A3.19}$$

for any A, $A \geq R$.

It follows directly from the definition of $\mathcal{D}_v$ that $\mathcal{D}_v g \in W^{1,1}_{loc}$, provided $g \in l^2(L_1)$ and moreover, one has the estimate

$$\|x^{-1/2}\frac{d}{dx}\mathcal{D}_v g\|_{l^2(L_1)} \leq C\|g\|_{l^2(L_1)}. \tag{A3.20}$$

(A3.16), (A3.19), (A3.20) imply that both $\mathcal{D}_v$ and $\mathcal{D}_v\mathcal{V}\mathcal{R}_v$ are compact operators in L_2. ∎

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